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THE

MATHEMATICAL GAZETTE.

EDITED BY

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WITH THE CO-OPERATION OF

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A. 2. a.; L¹. 10. a.]

PORISMATIC EQUATIONS.

Read before the Mathematical Association on January 21st, 1899.

(CONTINUED FROM PAGE 257.)

10. Given the equations:

$$\left.\begin{aligned} &u\cos\alpha\cos\beta+v\sin\alpha\sin\beta+w+u'(\sin\alpha+\sin\beta)\\ &\qquad+v'(\cos\alpha+\cos\beta)+w'\sin(\alpha+\beta)=0,\\ &u\cos\alpha\cos\gamma+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots=0,\\ &u\cos\beta\cos\gamma+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots=0,\end{aligned}\right\}\ldots\ldots\text{(B)}$$

let it be required to find the porismatic relation between the coefficients u, v, w, u', v', w', it being supposed that α, β, γ are all unequal and that no two of them differ by a multiple of 2π.

If we put
$$\cos\alpha+i\sin\alpha=x,$$
$$\cos\beta+i\sin\beta=y,$$
$$\cos\gamma+i\sin\gamma=z,$$

the first of the above equations (B) becomes

$$u(x+1/x)(y+1/y)-v(x-1/x)(y-1/y)+4w$$
$$+2(u'/i)(x-1/x+y-1/y)$$
$$+2v'(x+1/x+y+1/y)+2(w'/i)(xy-1/xy)=0,$$

or $\quad a/xy+bxy+c+f(x+y)+g(1/x+1/y)+h(x/y+y/x)=0,$

where $\quad a=u-v+2iw',\quad b=u-v-2iw',\quad c=4w,$

$\qquad f=2(v'-iu'),\quad g=2(v'+iu'),\quad h=u+v.$

The porismatic relation (x.) therefore becomes

$$(u+v)^2-\{(u-v)^2+4w'^2\}-4w(u+v)+4(u'^2+v'^2)=0,$$

or $$w'^2-u'^2-v'^2+wu+wv-uv=0.\ \ldots\ldots\ldots\ldots\text{(xi.)}$$

11. If the three relations (ix.) are similarly transformed and the real and imaginary parts of each equated to zero, it is found that

$$\left.\begin{aligned}
&(u+v)\,\Sigma\cos\alpha-(u-v)\cos(\alpha+\beta+\gamma)\\
&\qquad\qquad-2w'\sin(\alpha+\beta+\gamma)+2v'=0,\\
&(u+v)\,\Sigma\sin\alpha-(u-v)\sin(\alpha+\beta+\gamma)\\
&\qquad\qquad+2w'\cos(\alpha+\beta+\gamma)+2u'=0,\\
&(u+v)\,\Sigma\cos(\beta+\gamma)+2u'\sin(\alpha+\beta+\gamma)\\
&\qquad\qquad+2v'\cos(\alpha+\beta+\gamma)-(u-v)=0,\\
&(u+v)\,\Sigma\sin(\beta+\gamma)-2u'\cos(\alpha+\beta+\gamma)\\
&\qquad\qquad+2v'\sin(\alpha+\beta+\gamma)-2w'=0,\\
&2u'\Sigma\sin\alpha+2v'\Sigma\cos\alpha+(u-v)\,\Sigma\cos(\beta+\gamma)\\
&\qquad\qquad+2w'\Sigma\sin(\beta+\gamma)+4w-(u+v)=0,\\
&-2u'\Sigma\cos\alpha+2v'\Sigma\sin\alpha+(u-v)\,\Sigma\sin(\beta+\gamma)\\
&\qquad\qquad-2w'\Sigma\cos(\beta+\gamma)=0.
\end{aligned}\right\}\ldots\text{(xii.)}$$

It is however clear, *a priori*, that there are only two distinct relations between α, β, γ; and in fact the last four can be deduced from the first two.

The proof of the result (xi.) without imaginary quantities is rather long and intricate (Hobson's *Trigonometry*, pp. 87, 88). No mention is there made of the symmetrical relations (xii.).

12. In Nixon's *Trigonometry*, Chap. VIII., p. 163, No. 14, the following question is proposed:

"Show that the elimination of β between

$$x^2\cos\alpha\cos\beta+x(\sin\alpha+\sin\beta)+1=0,$$

and $$x^2\cos\beta\cos\gamma+x(\sin\beta+\sin\gamma)+1=0,$$

produces $$x^2\cos\gamma\cos\alpha+x(\sin\gamma+\sin\alpha)+1=0.$$

(*Ed. Times*, XLV., p. 123; Math. Tripos, '69; John's, Camb. '81.)"

Here $u=x^2,\ v=0,\ w=1,\ u'=x,\ v'=0,\ w'=0.$

Hence (xi.) is identically satisfied, so that the third equation is derivable from the other two. Moreover, the symmetrical relations between the variables are

$$\Sigma\cos\alpha=\cos(\alpha+\beta+\gamma)=\Sigma\sin(\beta+\gamma)/2x.$$

Similar questions in the same work are to be found on p. 155, No. 4; p. 166, No. 32; p. 167, No. 37; p. 361, No. 31; also in Hobson's *Trigonometry*, Chap. VI., Nos. 11, 12, 23, 32, 43, etc.

13. **The result for four variables may be briefly registered.**

Given the four equations:

$$\frac{a}{x_1x_2}+bx_1x_2+c+f(x_1+x_2)+g(1/x_1+1/x_2)+h(x_1/x_2+x_2/x_1)=0,$$

with similar equations involving (x_2, x_3), (x_3, x_4), (x_4, x_1), the relation between the constants is

$$abc+2fgh-af^2-bg^2-ch^2=0,$$

and the relation between the variables

$$\left.\begin{aligned}(x_1+x_3+f/b)(x_2+x_4+f/b)&=(f^2-bc)/b^2,\\ (x_1x_3-h/b)(x_2+x_4+f/b)&=(x_2x_4-h/b)(x_1+x_3+f/b)\\ &=(bg-fh)/b^2,\\ (x_1x_3-h/b)(x_2x_4-h/b)&=(h^2-ab)/b^2.\end{aligned}\right\}$$

14. Finally, the whole of the results of this note are true, *mutatis mutandis*, when a, b, c, f, g, h are no longer constants, but are symmetrical functions of the variables. R. F. DAVIS.

SOME CURIOSITIES IN DIVISION.

(CONTINUED FROM PAGE 208.)

Examples.

$$\text{(i.)}\ \frac{1}{7}=\frac{14}{100-2}$$

$$=\frac{\cdot 14}{1-\cdot 02}$$

$$\begin{array}{rr} = & \cdot 14{,}28{,}56{,}12 \\ + & \dot{1}\ \ 2 \\ \hline = & \cdot\dot{1}4\,28\,5\dot{7} \end{array}$$

$$\text{(ii.)}\ \frac{1}{17}=\frac{6}{100+2}$$

$$=\frac{6(100-2)}{10000-4}$$

$$=\frac{588}{10000-4}$$

$$=\cdot 0588\{1+\cdot 0004+(\cdot 0004)^2+\text{etc.}\}$$

$$\begin{array}{rr} = & \cdot 0588{,}2352{,}9408{,}7632{,}0528 \\ & 3\quad 15\quad\ 60 \end{array}$$

$$=\cdot\dot{0}58823529411764\dot{7}.$$

(iii.) $\frac{1}{13} = \frac{3}{39}$ or $\frac{1}{13} = \frac{7}{91}$ or $\frac{1}{13} = \frac{23}{299}$

$$= \frac{\cdot 075}{1 - \frac{1}{40}} \qquad = \frac{\cdot 07}{1 - \cdot 09} \qquad = \frac{\cdot 23}{3(1 - \frac{1}{300})}$$

$$\begin{array}{rl} = & \cdot 075923076 \\ + & \underline{\quad 1 \quad} \\ & \cdot\dot{0}7692\dot{3} \end{array} \qquad \begin{array}{r} = \cdot 07,63,67,03 \\ 5,51,59 \\ \underline{4,41} \\ \cdot\dot{0}7\;69\;2\dot{3} \end{array} \qquad \begin{array}{l} = \frac{1}{3}\{\cdot 23\dot{0}7692\dot{3}\} \\ = \cdot\dot{0}7692\dot{3}\,^{*} \end{array}$$

$$\begin{aligned} \frac{1}{23} &= \frac{13}{299} \\ &= \frac{\cdot 13}{3(1 - \frac{1}{300})} \\ &= \tfrac{1}{3}(\cdot 13\dot{0}43478260869565217391\dot{3}). \end{aligned}$$

the period marked with recurring points being that required.

$$\begin{aligned} \frac{1}{31} &= \frac{30 - 1}{900 - 1} \\ &= \frac{\cdot 29}{9\left(1 - \frac{1}{900}\right)} \\ &= \frac{1}{9}\{\cdot 29\dot{0}3225806451612\dot{9}\}. \end{aligned}$$

$$\begin{aligned} \frac{1}{43} &= \frac{7}{301} \\ &= \frac{7 \times 299}{90000 - 1} \\ & \frac{\cdot 2093}{(\frac{1}{90000})9 - 1} \\ &= \frac{1}{9}(\cdot 2093\dot{0}23255813953488372093\dot{3}). \end{aligned}$$

Some of the devices here used enable us to get many recurring decimals by continuous multiplication. EDWARD M. LANGLEY.

THEOREMS CONNECTED WITH INVERSION.

The square of the tangent from a point P to a circle S, divided by the diameter of S, will, in what follows, be denoted by (PS).

If S open out into a straight line, it is easily seen that (PS) equals in the limit the perpendicular distance of P from this line; accordingly the length of the perpendicular from a point P on a straight line S will be denoted by the *same symbol* (PS).

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* Here the successive digits are obtained by continued division by 3. Compare the work for $\frac{1}{23}$.

PROPERTIES OF (PS).

(*a*) If S (a straight line or circle) be inverted with regard to any point P, the length of the diameter of its inverse can be shown to be $k^2/(PS)$, where k is the radius of inversion.

(*b*) If a point-circle at P of infinitesimal radius r intersect S at an angle $\widehat{PS}$; then $(PS) = -r\cos\widehat{PS}$.

For $r^2+R^2-c^2=2Rr\cos\widehat{PS}$, where R is the radius of S and c the distance of its centre from P. Retaining the first power of r only, the property follows at once.

(*c*) *If P and S be inverted from O, then will $(PS)/PO$ be unchanged.**

To prove this, $\qquad (PS) = -r\cos\widehat{PS}$.

Similarly, $\qquad (P_1S_1) = -r_1\cos\widehat{P_1S_1}$,

where suffixes refer to the inverse figures. Now

$$r/r_1 = PO/P_1O, \text{ and } \widehat{PS} = \widehat{P_1S_1};$$

whence $\qquad (PS)/PO = (P_1S_1)/P_1O$.

For example, invert the following:

Any point P on the bisector of an angle $\widehat{LM}$ is equidistant from L and M.

Here $(PL)=(PM)$;

$$\therefore \ (PL)/PO=(PM)/PO,$$

giving $\quad (P_1L_1)/P_1O=(P_1M_1)/P_1O$, or $(P_1L_1)=(P_1M_1)$.

Now L_1M_1 are *circles* meeting at O, and P_1 lies on a circle passing through their common points and meeting them at equal angles; also it follows from the last equation that *the squares of the tangents from P_1 to these circles are in the ratio of their diameters.*

(*d*) A variable circle S, meeting two fixed circles at angles whose cosines have a given ratio, is known to cut a fixed circle orthogonally. Let the centres of these circles, *which we may suppose point-circles*, be A, B; then by (*b*), $(AS):(BS)$ is constant. It follows that *a circle which varies so that the lengths of the tangents to it from two fixed points A, B are in a constant ratio cuts a fixed circle orthogonally.*

(*e*) A circle which cuts two circles at given angles envelopes a pair of circles. Apply this to the point circles in (*d*) and we find that S *envelopes a pair of circles when two fixed points A, B can be found such that (AS) and (BS) are separately constant.*

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* (*c*) is equivalent to Q. 14265, *Educational Times*, by Mr. R. F. Davis, and suggested this method to the writer.

By the aid of this principle, choosing as base points MN (*Gazette*, p. 257), the extension of Feuerbach's Theorem given there can be established.

(f) Let S denote the circumcircle of a triangle.

If P coincide in turn with the circumcentre, incentre, and excentres, the corresponding values of (PS) are

$$-\tfrac{1}{2}R,\ -r,\ r_1,\ r_2,\ r_3.$$

To prove the second: $R^2-d^2=2Rr$ by Euler's Theorem; where d is the distance of the circumcentre from incentre. Hence

$$(PS)\equiv(d^2-R^2)/2R=-r.$$

It is obvious that we may substitute "sphere" for circle, and "plane" for straight line in (a), (b), (c), (d), (e). C. E. M'VICKER.

OBITUARY.

The Mathematical Association has lost an efficient officer in the sudden death of Samuel Oliver Roberts, who had been one of our honorary secretaries since 1897. His mathematical tastes were undoubtedly inherited from his father, Mr. Samuel Roberts, F.R.S., whom he has predeceased. Mr. S. O. Roberts went to Cambridge as a scholar of St. John's College in 1879, and was seventh wrangler in 1882. At Cambridge he made many fast friends, some of whom he constantly revisited there, thus ever retaining a close touch with his Alma-Mater. After leaving Cambridge he was for three or four years at the Royal Grammar School, Newcastle-on-Tyne. Thence he was appointed in 1888 to the second mathematical mastership at the Merchant Taylors' School, and retained the post till the time of his death. He was an active member of the Physical Society and the London Mathematical Society, and an ardent student of modern languages and modern history. But it was to his school work that he devoted his enthusiasm and abundant energies. He was pre-eminent as a teacher, and often took delight in contrasting the mathematical teaching which he himself experienced at school with that of the present day, to the glorification of the latter. He also took a great interest in school sports, and won the admiration of his pupils by his mastery of chess. He was the admiration of all his friends for his strenuous devotion to work. He laboured hard for his school, possibly so hard as to have an adverse influence on his health, though we would fain believe not. He had been suffering from an apparently slight indisposition for some little time; this suddenly took a serious form just before Whitsuntide, and ended fatally on the evening of May 31st. A more detailed account is given in the school magazine, *The Taylorian*, Vol. XXI., No. 6, July 1899.

MATHEMATICAL NOTES.

73. *On the circles touching three given tangential circles.*

If O, O_1, O_2 be the centres of the three given circles; r, r_1, r_2 their radii; I the centre of one of the circles touching each of the three; θ, θ_1, θ_2 the angles $O'IO''$, etc., and R the radius of the circle, centre I, then the sides of the triangle $O'IO''$ are r_1+R, r_2+R, r_1+r_2.

Then $\sin^2\dfrac{\theta}{2}=r_1r_2/(r_1+R)(r_2+R)$, etc.

But $$\Sigma \sin^4\frac{\theta}{2} - 2\Sigma \sin^2\frac{\theta}{2}\sin^2\frac{\theta_1}{2} + 4\Pi \sin^2\frac{\theta}{2} = 0 ; \quad \because \quad \Sigma\theta = 2\pi.$$

Substituting the values of $\sin^2\frac{\theta}{2}$, etc., we have a quadratic for R.

If O_1, O_2, for instance, touch each other externally and O internally, the value of R is obtained from the quadratic by writing $-r_1$, $-r_2$ for r_1, r_2.

E. N. Barisien.

74. *Asymptotes in polar coordinates.*

If $u \equiv 1/r = f(\theta)$ is the equation of a plane curve, it is possible to approximate as follows to its form in the neighbourhood of $\theta = a$, where $f(a) = 0$.

For, if $\theta = a + \phi$ and ϕ is a small quantity of the first order of infinitesimals, then by Taylor's Theorem,

$$u = f(a+\phi) = \phi f'(a) + \frac{\phi^2}{2!}f''(a) + \dots .$$

Thus, if $f'(a)$ is not zero or infinite, $u = \phi f'(a)$ to the first order. This is equivalent to $u = \sin\phi f'(a)$ to the same order, and leads to the equation of the linear asymptote in the form $r\sin(\theta - a) = 1/f'(a)$.

If $f'(a)$ vanishes, and $f''(a)$ is neither zero nor infinite, then $u = \frac{1}{2}\phi^2 f''(a)$ to the same order $= 2\sin^2\frac{\phi}{2}f''(a)$ to the same order. We thus have the equation of the parabolic asymptote, viz., $r\{1 - \cos(\theta - a)\} = 1/f''(a)$. H. T. Gerrans.

EXAMINATION QUESTIONS AND PROBLEMS.

Our readers are earnestly asked to help in making this section of the Gazette *attractive by sending either original or selected problems.*

Solutions should be sent within three months of the date of publication. They should be written clearly on one side of the paper. Contractions not intended for printing should be avoided. Figures should be drawn with the greatest care on as small a scale as possible, and on a separate sheet.

The question need not be re-written, but the number should precede every solution.

The source of problems when not otherwise indicated is shown by—C. (Cambridge), O. (Oxford), D. (Dublin), W. (Woolwich), Sc. (Science and Art Department), etc.

320. [L[1]. 16. b.] (*a*) A fixed point P is joined to a point M on a conic, and a circle is described on PM as diameter. Find the envelope of the circles as M moves on the conic, and discuss the cases in which P is (1°) at the centre; (2°) a focus; (3°) a vertex.

[L[1]. 4. c.] (*b*) Given two conics C, C', a tangent to C cuts C' in A and B. Find the locus of the intersections of the tangents to C from A and B. E. N. Barisien.

321. [J. 1. c.] If H_r denote the sum of the homogeneous products of r dimensions of n quantities a, β, γ, ... then any function of their differences satisfies

$$n\frac{du}{dH_1} + (n+1)H_1\frac{du}{dH_2} + \dots = 0.$$

E. P. Barrett.

322. [L². 2. d.] A plane cuts a cone, vertex O, in an ellipse, focus S. The tangent to the ellipse at P meets the directrix in Q. Show that Q is the pole with respect to the focal sphere of the plane OSP. J. A. BRERETON.

323. [K. 4.] Construct an equilateral triangle, given the centres of its in-squares. W. S. COONEY.

324. [L¹. 3. c.] CP, CD, conjugate radii of an ellipse, are each multiplied by $\frac{b}{a}$ and turned through $+90°$ into the lines CU, CL respectively. If CP be in the quadrant ACB, show that DU is parallel to CA, and $PL=a+b$. R. W. GENESE.

325. [L¹. 7. d.] (a) Given a point on a rectangular hyperbola and a focus, show that the second focus lies on a Limaçon. If two points are given on a conic, and one focus, the second focus lies on an ellipse.

(b) Given that a conic of constant minor axis is inscribed in a triangle, the foci lie on a cubic. W. J. GREENSTREET.

326. [J. 2. A.] A product of two given numbers is tested by "casting out the nines," and the result fails to show that the product is wrong. What is the chance of its being right? J. F. HUDSON.

327. [B. 2. a.] Show that there exist linear functions u, v of x, y, z, which will render

$$u\begin{vmatrix} x^2, & y^2, & z^2 \\ xx', & yy', & zz' \\ \alpha^2, & \beta^2, & \gamma^2 \end{vmatrix} + v\begin{vmatrix} x^2, & y^2, & z^2 \\ xx'', & yy'', & zz'' \\ \alpha^2, & \beta^2, & \gamma^2 \end{vmatrix} \text{ divisible by } \begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ x'', & y'', & z'' \end{vmatrix};$$

and determine u, v. W. J. JOHNSTON.

328. [K. 20. e.] In a triangle ABC prove that

$$\Sigma \sin 2A[R\cos(B-C)-2r]^2 = 2(\Pi \sin A)(R-2r)^2.$$

H. M. LESLIE.

329. [D. 2. b. β.] If

$$C_n = \frac{\cos n\Sigma\alpha}{\Pi\sin(\beta+\gamma)} + \Sigma\frac{\cos n(\beta+\gamma-\alpha)}{\sin(\beta+\gamma)\sin(\beta-\alpha)\sin(\gamma-\alpha)},$$

then $c_0=c_1=c_2=0$; $c_3=-32\Pi\sin\alpha$; $c_4=-32\Pi\sin 2\alpha$.

F. S. MACAULAY.

330. [I. 2. b.] If n be an integer prime to 2 and 3, n^2-1 is divisible by 24. If n be an integer prime to 2, 3, and 5, $(n^2-1)^2(n^2-4)$ is divisible by 8640. P. A. MACMAHON.

331. [L¹. 5. a.] The tangents at the end of a focal chord PQ to a conic meet in T. The normal at P meets the axis in G and TT in L. Show that PL varies directly as PG. H. G. MAYO.

332. [R. 1. d.] If P, P' be two successive positions of a point moving uniformly in a straight line, and Q, Q' two corresponding positions of a second point moving in like manner in another straight line, prove the following construction for the straight line which joins the moving points at the instant when they are collinear with a fixed point R. Draw the complete quadrilateral whose sides are PP', QQ', PQ, $P'Q'$, and let S be the point of intersection of the circles about its triangles, and let L be the pedal line of one of these triangles for the point S. On RS as diameter draw a circle cutting L in M and N. Each of the lines RM, RN is a solution of the problem.

Modify the above construction for the case in which R is not a fixed point, but one moving uniformly in a straight line from R to R' while the first point moves from P to P'.

R. F. MUIRHEAD.

333. [K. 2. b.] Through the vertex of any triangle can be drawn a circle touching the circumcircle, the incircle, and the excircle opposite that vertex: also a second circle touching the circumcircle and the remaining excircles; and the sum of the diameters of these two circles equals that of the circumcircle.

C. E. M'VICKER.

334. [K. 2. b. e.] H is the orthocentre of a triangle ABC. Six tangents are drawn to the incircle perpendicular to BC, CA, AB and meeting them in the pairs of points L and L', M and M', N and N' respectively. Three tangents are drawn to the incircle perpendicular to AH, BH, CH and meeting these lines in P, Q, R respectively. The connectors LH, $L'H$ meet the circle having AH as diameter in l, l' respectively: MH, $M'H$ meet the circle having BH as diameter in m, m' respectively; and NH, $N'H$ meet the circle having CH as diameter in n, n' respectively. Prove that the following sets of points are concyclic: $HQLL'R$, $HRMM'P$, $HPNN'Q$, $LL'MM'NN'$, $ll'mm'nn'$. J. A. THIRD.

335. [I. 1.] The minute hand of a clock in descending slips down at a constant rate of 1 minute space per 30 true minutes, and in ascending slips back at the same rate. It is set right at noon. Find the next two and the last occasions when it shows true time, and prove that in $37\frac{1}{2}$ days it is 1 hr. late for the first time. T. P. THOMPSON.

336. [L¹. 5. d.] Find by polar coordinates the locus of the intersection of two normals, the inclinations of which to the axis are ϕ, 2ϕ. A. S. TURNER.

337. [A. 1. c.] If the greatest coefficient in the expansion of $(1+x)^n$ be a multiple of n and of $(n-2)$, then n must be of the form $6p+1$, or $6p+3$. H. R. TYLER.

338. [D. 2. b. β.] If $\theta+\phi+\psi=4\pi$,

$$\sum_0^\infty\left[(-1)^r\frac{\Sigma\theta^{2r}}{\lfloor 2r}\right]+1$$

$$=4\prod_0^\infty\left[1-\frac{\Sigma\theta^2}{\pi^2}\cdot\frac{1}{(2r+1)^2}+\frac{\Sigma\theta^2\phi^2}{\pi^4}\cdot\frac{1}{(2r+1)^4}-\frac{\theta^2\phi^2\psi^2}{\pi^6}\cdot\frac{1}{(2r+1)^6}\right].$$

L. WALLIS.

339. [K. 2. c.] D, E, F being the points whose pedal lines pass through the nine-point centre, show that in the circle ABC the arcs AD, BE, CF are one-third of the arcs AA', BB', CC' cut off by chords parallel to BC, CA, AB; and that the triangle DEF is equilateral. C. E. YOUNGMAN.

SOLUTIONS.

A great number of Solutions are in hand, and will be published as sufficient space is available. Solutions of the following are not yet to hand—
171, 252, 258, 265, 266, 271, 274-6, 278-9, 283, 285, 287, 291, 299, 303, 306-7, 312-13, 317, 319*a*.

167. [R. 9. b.] *Three equal balls (coefficient of elasticity e) are at rest in a straight line on a horizontal plane. If A is projected towards B, show there will be at least five impacts if* $e+e^{-1}>4+2\sqrt{5}$. PET. AND SID. '96.

Solution by E. LL. TANNER, J. L. THOMAS.

Suppose A to be projected towards B with velocity v, the velocities after the first impact are given by

$$v_A+v_B=v,$$

$$v_A-v_B=V-ev.$$

$$v_A=\frac{v(1-e)}{2},\quad v_B=\frac{v(1+e)}{2}.$$

B then goes on and strikes C with velocity $\frac{v(1+e)}{2}$, and therefore after two impacts the velocities of the balls are

$$\frac{v(1-e)}{2},\ \frac{v(1-e^2)}{4},\ \frac{v(1+2e+e^2)}{4}.$$

The velocity of B is $\frac{1+e}{2}$ times that of A, *i.e.* A is moving faster than B, and there must be a third impact. Their velocities after the third impact are given by

$$v'_A+v'_B=\frac{v}{4}(3-2e-e^2),$$

$$v'_A-v'_B=\frac{v}{4}(-e+2e^2-e^3).$$

$$v'_A=\frac{v}{8}(3-3e+e^2-e^3),$$

$$v'_B=\frac{v}{8}(3-e-3e^2+e^3).$$

There will be a fourth impact if

$$3-e-3e^2+e^3>2+4e+2e^2.$$

$$\therefore\ (3-e)(1+e)(1-e)>2(1+e)^2,$$

$$\therefore\ 3-4e+e^2>2+2e,$$

$$1-6e+e^2>0.$$

Now the roots of $1-6e+e^2=0$ are $3\pm2\sqrt{2}$, and therefore e must not lie between these.

Now e is <1; and therefore e must be $<3-2\sqrt{2}$ (for $3+2\sqrt{2}>1$).

The velocities after the fourth impact are given by

$$v''_B=v''_C=\frac{v}{8}(1+e)(5-2e+e^2),$$

$$v''_B-v''_C=\frac{v}{8}(1+e)(-e+6e^2-e^3).$$

$$v''_B=\frac{v}{16}(1+e)(5-3e+7e^2-e^3).$$

There will be a fifth impact if $\frac{v}{8}(3-3e+e^2-e^3)$ be greater than this.

The condition is

$$2(3-3e+e^2-e^3)>5+2e+4e^2+6e^3-e^4,$$

$$6-6e+2e^2-2e^3>5+2e+4e^2+6e^3-e^4,$$

$$1-8e-2e^2-8e^3+e^4>0,\quad \left(e^2+\frac{1}{e^2}\right)-8\left(e+\frac{1}{e}\right)-2>0,\quad \left(e+\frac{1}{e}\right)^2-8\left(e+\frac{1}{e}\right)-4>0.$$

The roots of $x^2-8x-4=0$ are $4\pm2\sqrt{5}$, $\therefore$ there will be a fifth impact if

$$(e+e^{-1})>4+2\sqrt{5}.$$

267. [A. 3. b.] *If the roots of the equation*

$$a_0x^n-na_1x^{n-1}+\frac{n(n-1)}{1.2}a_2x^{n-2}-\ldots+(-1)^na_n=0$$

are all positive, show that $a_ra_{n-r}>a_0a_n$ *for all values of* r *between* 1 *and* $n-1$ *inclusive, unless the roots are all equal.* A. Lodge.

Solution by C. E. M'Vicker.

The arithmetic mean of any number of positive and unequal quantities is greater than their geometric mean, *i.e.*

$$p_1+p_2+\ldots+p_m>m(p_1p_2\ldots p_m)^{\frac{1}{m}}.$$

Applying the same theorem to their reciprocals, we find on multiplication

$$\Sigma(p).\Sigma\left(\frac{1}{p}\right)>m^2.$$

Now taking p to denote a product, r together, of the roots of given equation

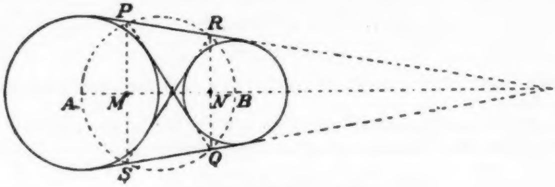
$$\Sigma(p)={}^nC_r.\,a_r/a_0,$$

$$\Sigma(1/p)={}^nC_r.\,a_{n-r}/a_n.$$

Also m, the number of terms in each Σ, is ${}^nC_r$,

$$\therefore\ ({}^nC_r.\,a_r/a_0)({}^nC_r.\,a_{n-r}/a_n)>({}^nC_r)^2 \text{ or } a_ra_{n-r}>a_0a_n.$$

268. [K. 11. d.] *The distance between the pair of inverse points common to two circles equals the product of the direct and transverse common tangents, real or imaginary, divided by the distance between the centres.* C. E. M'VICKER.



Solution by PROPOSER.

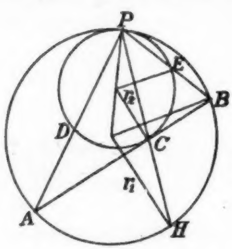
Let A, B be the centres of the given circles. PA, PB bisect the angle between the common tangents from P, and are therefore at right angles; hence P lies on the circle on AB as diameter.

This circle also passes through Q, R, S, the remaining points of intersection of the common tangents.

Join PS, RQ meeting the line of centres in M, N. From the harmonic properties of the complete symmetric quadrilateral $PQRS$, M, N are inverses with respect to each circle.

Again, in the triangle PQR, $PQ.PR$=rect. under altitude and diameter of circumcircle$=AB.MN$, which proves the theorem; for it is easily seen that PQ, PR are respectively equal in length to the direct and transverse common tangents.

273. [K. 11. d.]. *Two circles touch internally at P, and a chord AB of the outer touches the inner at C. If PA, PB cut the inner circle at D, E, show that $PD.PE$ is to PC^2 as the radius of the inner circle is to the radius of the outer.* J. V. THOMAS.



SOLUTION BY H. G. MAYO; W. S. COONEY; H. P. KNAPTON.

The similar triangles PAC, PCE give

$$PA.PE=PC^2.$$

Similarly, $$PB.PD=PC^2;$$

$$\therefore\ PD:PA=PE:PB=r_1:r_2;$$

$$\therefore\ PD.PE:PA.PE=PD.PE:PC^2;$$

$$\therefore \text{ etc.}$$

277. [M. 6. n.] *Show that the maximum value of the perpendicular from the pole of $r=a(1-\cos\theta)$ on the normal is $4\sqrt{3}a/9$.* TRIN. C. 1890.

Solution by C. E. M'VICKER.

The equation $r=a+a\cos(\pi-\theta)$ is that of the pedal of a circle of radius a with respect to a point on its circumference.

Let O be the pole, C the centre of the circle. Take CO produced, as initial radius. Draw OY perpendicular to the tangent at any point P; locus of Y is given curve.

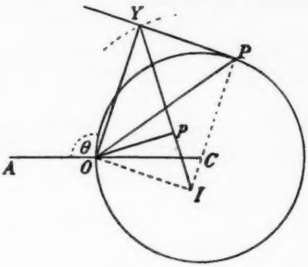
By a known theorem, normal at Y to pedal passes through I, the opposite vertex of the rectangle having OY, YP for adjacent sides. If $Op\equiv p$ be at right angles to IY, then maximum value of p is required.

From right-angled triangle OIY,

$$1/p^2 = 1/OI^2 + 1/OY^2; \quad \therefore \quad a^2/p^2 = 1/\sin^2\theta + 1/(1-\cos\theta)^2,$$

which may be written

$$4p^2/a^2 = (1-\cos\theta)(2+2\cos\theta)(1-\cos\theta).$$



The factors on right side are positive; also their *sum* is independent of θ; hence when they are equal, *i.e.* when $\cos\theta = -\frac{1}{3}$, p is a maximum. In this case

$$4p^2/a^2 = \left(\frac{4}{3}\right)^3;$$

$$\therefore \quad p = \frac{4a\sqrt{3}}{9}.$$

280. [K. 16. a.]. *The angles of one face of the in-cube of a sphere meet the surface at A, B, C, D. If P is any point on the sphere* $3\Sigma\cos^2 PA = 4$.

W. S. Cooney.

Solution by C. Williams.

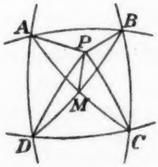
Take O the centre of the sphere as origin, and Ox perpendicular to the face $ABCD$. Then the direction cosines of OA, OB, OC, OD are $\frac{1}{\sqrt{3}}$, $\pm\frac{1}{\sqrt{3}}$, $\pm\frac{1}{\sqrt{3}}$. Therefore if λ, μ, ν be the direction cosines of OP,

$$\Sigma\cos^2 PA = \Sigma\left(\frac{1}{\sqrt{3}}\lambda \pm \frac{1}{\sqrt{3}}\mu \pm \frac{1}{\sqrt{3}}\nu\right)^2 = \tfrac{4}{3}(\lambda^2+\mu^2+\nu^2) = \tfrac{4}{3}.$$

Solution by W. S. Cooney.

Diagonals of quadrilateral bisect each other at right angles at M. Let

$$\angle PMA = \theta,\ AM = \phi,\ PM = x,$$



then $\quad \cos^2 PA + \cos^2 PC = 2\cos^2\phi\cos^2 x + 2\sin^2\phi\sin^2 x\cos^2\theta,$

also $\quad \cos^2 PB + \cos^2 PD = 2\cos^2\phi\cos^2 x + 2\sin^2\phi\sin^2 x\sin^2\theta;$

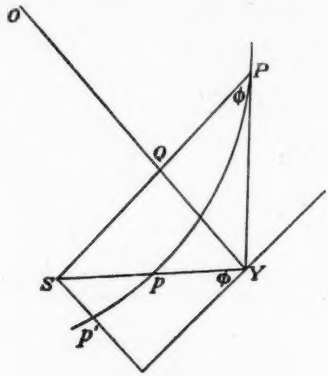
$$\therefore \quad \Sigma\cos^2 PA = 4\cos^2\phi\cos^2 x + 2\sin^2\phi\sin^2 x;$$

but $\quad \cos AC = -\frac{1}{3}; \quad \therefore \quad 2\cos^2 x - 1 = -\frac{1}{3};$

$$\therefore \quad \cos^2 x = \tfrac{1}{3}; \quad \sin^2 x = \tfrac{2}{3};$$

$$\therefore \quad \Sigma\cos^2 PA = \tfrac{4}{3}\cos^2\phi + \tfrac{4}{3}\sin^2\phi = \tfrac{4}{3}; \quad \therefore \text{ etc.}$$

281. [O. 2. e.]. *SY is the perpendicular from the pole on the tangent at P to a given curve. Bisect SP in Q; join YQ and produce it to O so that QO : YO = chd. of curv. at P through S : 4SP. Then O is the centre of curvature at Y to the pedal.* R. F. DAVIS.



Solution by A. POOLE.

The normal at Y to the pedal obviously passes through the middle point of SP;
and if $$QO : YO = 2\rho \sin\phi : 4r;$$

$$\frac{r}{2} : YO = 4r - 2\rho\sin\phi : 4r.$$

$$\therefore\ YO = \frac{r^2}{2r - \rho\sin\phi} = \frac{r^2}{2r - p\dfrac{dr}{dp}} = \frac{pr^2}{2pr - p^2\dfrac{dr}{dp}} = \frac{p}{\dfrac{d}{dp}\left(\dfrac{p^2}{r}\right)}$$

$$= p \cdot \frac{dp}{dp'} = \text{radius of curvature at } Y,$$

for p' (for pedal curve) $= \dfrac{p^2}{r}$.

For the Proposer's solution by infinitesimals see *Educational Times Reprint*, vol. lxx., p. 91.

282. *Two circles are described with the foci of the ellipse as centres, so that the sum of the squares of their radii is $4a^2 - 2b^2$ (usual notation). The circles cut any tangent to the ellipse in a harmonic range.* (B.A., London.)
F. A. FIELD (*corrected*).

Solution by J. F. HUDSON.

Let the circles cut the tangent in PQ, pq respectively.
Draw SM, HN perpendicular to the tangent.
Then M, N are the mid-points of PQ and pq respectively.
Let R, r be the radii of the circles.

Then $$2SM \cdot HN = SM^2 + HN^2 - (SM - HN)^2,$$

or $$2b^2 = R^2 + r^2 - MQ^2 - pN^2 - SH^2 + MN^2;$$

but $$MN^2 - pN^2 - MQ^2 = Mp \cdot Mq - MQ^2 = 0, \text{ if } (PpQq) = -1.$$

Hence if $(PpQq)=-1$, we must have

$$2b^2=R^2+r^2-SH^2,$$

or $$R^2+r^2=2b^2+4(a^2-b^2)=4a^2-2b^2.$$

284. [K. 2. a.]. *LMN is the Simson line of the triangle ABC with respect to P, on the side of BC remote from A. PA, PB, PC subtend angles A′, B′, C′ at the circumference. Show that*

(1) $\sin^2 A=\lambda u/vw$, *etc., where* u, v, w *denote* $MN \operatorname{cosec} A$, $NL \operatorname{cosec} B$, $LM \operatorname{cosec} C$, *and* $\lambda\left(-\dfrac{\sin A}{u}+\dfrac{\sin B}{v}+\dfrac{\sin C}{w}\right)=\Pi \sin A$; (Trip., 1891.)

(2) $PL\,.\,PA=PM\,.\,PB=PN\,.\,PC$;

(3) $$\sin^2 A\,.\,\sin^2 A'\,.\,\frac{LM\,.\,LN}{MN}=\sin^2 B\,.\,\sin^2 B'\,.\,\frac{MN\,.\,ML}{NL}$$

$$=\sin^2 C\,.\,\sin^2 C'\,.\,\frac{NL\,.\,NM}{LM}.$$ W. J. Greenstreet.

Solution by W. S. Cooney, H. A. Saunders, C. E. Youngman.

Since $AMPN$ is a circle with AP for diameter, $MN=AP\sin A$. Hence u, v, w denote PA, PB, PC; u being negative, because MN is measured in the opposite direction to NL and LM. Let CC_1 be a diameter of the circle ABC; then from similar triangles PBL and CC_1P,

$$PB:BL=CC_1:CP\,;\ \therefore\ PB\,.\,PC=2R\,.\,PL,$$

i.e. $vw=2R\alpha$ (α, β, γ denoting PL, PM, PN).

Hence $u\alpha=v\beta=w\gamma=uvw/2R$, proving No. 2.

And the equation for λ becomes

$$\lambda 2R(\alpha\sin A+\beta\sin B+\gamma\sin C)=uvw\sin A\sin B\sin C,$$

giving $$\lambda=uvw/4R^2\,;$$

$\therefore$ the relation $\sin^2 A'=\lambda u/vw$, ...(1)

becomes $\sin^2 A'=u^2/4R^2$ or $2R\sin A'=AP$, which is evidently true.

The relation (3) requires $\sin^2 A\sin^2 A'/MN^2=$ etc.,

or $$\sin^2 A'/PA^2=\sin^2 B'/PB^2=\sin^2 C'/PC^2,$$

which is true, since each $=1/(4R^2)$.

Mr. Youngman remarks that if we determine the angles A', B', C' by measuring the arcs PA, PB, PC on which they stand all in the same direction, we have for any position of P, $\dfrac{BL}{LC}=\dfrac{\cot PBC}{\cot PCB}=-\dfrac{\cot C'}{\cot B'}$. Hence the areal equation of LMN may be written

$$\alpha\tan A'+\beta\tan B'+\gamma\tan C'=0.$$

286. [D. 6. b. γ.]. $\Sigma \sin^{-1} bc/\sqrt{(a^2+b^2)(a^2+c^2)}=\pi/2.$ A. Lodge.

Solution by R. P. Knapton, H. A. Saunders, C. Williams.

If $$\sin\theta_1=bc/\sqrt{(a^2+b^2)(a^2+c^2)},$$

then $$\tan\theta_1=bc/a\sqrt{a^2+b^2+c^2}\,;$$

$$\therefore\ \Sigma\tan\theta_1\tan\theta_2=1\,;\ \therefore\ \theta_1+\theta_2+\theta_3=\frac{\pi}{2}.$$

288. [J. 1. d.] *A, B, C, D are four boxes arranged in a circle, and n letters are divided into four packets, each of which may contain any number (zero included) of letters. A packet is placed in each box so that no box contains more letters than are in box A or fewer letters than are in box C. Show that the number of arrangements is given by the coefficient of x^n in the expansion of* $(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-1}$. P. A. MACMAHON.

Solution by J. C. M. GARNETT.

The required number is the coefficient of x^n in

I. $(1+x+x^2+\dots)(1+x+x^2+\dots)(1+x+x^2+\dots)(1+x+x^2+\dots)$,
after omitting certain terms as explained below;

II. or in $(1+x+x^2+\dots)[(1+x+x^2+\dots)(1+x+x^2+\dots).1$
$+(x+x^2+\dots)(x+x^2+\dots).x+(x^2+x^3+\dots)(x^2+x^3+\dots).x^2+\dots]$,
omitting certain terms;

III. or in $1.1^2+x[(1+x)(1+x)+x^3.1.1]+\dots+x^r[(1+x+x^2+\dots+x^r)^2$
$+x^3(1+x+x^2+\dots+x^{r-1})^2+x^6(1+x+x^2+\dots+x^{r-2})^2+\dots]+\dots$,

or in $(1+x^4+x^8+\dots)[1^2+x(1+x)^2+\dots+x^r(1+x+x^2+\dots+x^r)^2+\dots]$,

or in $\dfrac{1}{(1-x^4)(1-x)^2}[(1-x)^2+x(1-x^2)^2+\dots+x^r(1-x^{r+1})^2+\dots]$,

or in $\dfrac{1}{(1-x^4)(1-x)^2}[(1+x+x^2+\dots+x^r+\dots)$
$-2x(1+x^2+x^4+\dots+x^{2r}+\dots)+x^2(1+x^3+\dots+x^{3r}+\dots)]$,

or in $\dfrac{1}{(1-x^4)(1-x)^2}\left(\dfrac{1}{1-x}-\dfrac{2x}{1-x^2}+\dfrac{x^2}{1-x^3}\right)$,

i.e. in $\dfrac{1}{(1-x)(1-x^2)^2(1-x^3)}$.

Each retained term in the product I. is such that the index of the power of x, taken from either of the middle brackets, must not be less than that taken from the fourth bracket, nor more than that taken from the first. The terms in I. not satisfying the first condition are rejected in II., and those in II. not satisfying the second condition are rejected in III. Thus III. consists of the terms of I. satisfying both conditions.

289. [I. 17. a.]. *Find an indefinite number of rational right-angled triangles whose legs differ by unity.* [*Use* $(2a+b+2c)^2+(a+2b+2c)^2=(2a+2b+3c)^2$ *if* $a^2+b^2=c^2$.] ARTEMAS MARTIN.

Solution by PROPOSER.

Let $a+x$, $b+x$, and $nx-c$ be the sides of a rational right-angled triangle, in which the values of a, b, c are such that $a^2+b^2=c^2$; then

$$(a+x)^2+(b+x)^2=(nx-c)^2.$$

Expanding, cancelling, uniting like terms, and transposing,

$$(n^2-2)x=2a+2b+2nc\text{ ; whence } x=\frac{2(a+b)+2nc}{n^2-2}.$$

Take $n=2$, then $x=a+b+2c$;

$$\begin{aligned}\therefore\ a+x&=2a+b+2c=2(a+c)+b,\\ b+x&=a+2b+2c=2(b+c)+a,\\ nx-c&=2a+2b+3c=2(a+b+c)+c;\end{aligned}$$

$$\therefore\ (2a+b+2c)^2+(a+2b+2c)^2=(2a+2b+3c)^2.$$

If $a=3$, $b=4$, $c=5$, then $20^2+21^2=29^2$.

If $a=20$, $b=21$, $c=29$, then $119^2+120^2=169^2$.

290. [K. 11. e.]. *A circle is inscribed in a quadrant of a circle, and a second circle is drawn touching this circle, the quadrant, and one radius of the quadrant. Prove that the radius of the second is $\frac{1}{7}$th of the distance of its centre from the other radius of the quadrant.* C. E. M'Vicker.

Solution by Proposer.

Taking two equal orthogonal circles, describe a circle touching both and also touching one of their common tangents.

It is easy to prove that its radius is one-eighth of the radius of either, and hence one-seventh of the distance of its centre from the line of centres.

The ratio of the radius of a circle to the distance of its centre from any line is clearly the same as the corresponding ratio for its inverse, provided that the origin is taken *on* the line.

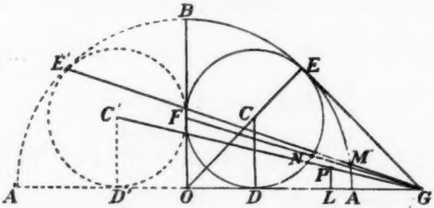
Take as origin a point in which one of the equal circles is cut by their line of centres, and invert the second circle into itself. According to the origin chosen, either the required property follows or a corresponding property for a circle touching a quadrant, one radius of the quadrant, and a circle *escribed* to the quadrant.

[The above problem was taken from Casey's *Sequel to Euclid.*]

Solution by C. E. Youngman, W. S. Cooney.

Let OAB be the quadrant, with centre O and radius R; D, E, F its points of contact with the inscribed circle, which has centre C and radius r. Draw EG the common tangent at E to cut OA at G. It is easily seen that $GD=GE=R$, and $\therefore$ $AG=OD=r$, and also that

$$R=r+r\sqrt{2}, \text{ and } r=R\sqrt{2}-R.$$



Let P be the centre of the second circle, and L, M, N its points of contact with OA, the arc AB, and the circle DEF. Draw the circle $D'E'F'$ (centre C') symmetrical to DEF on the other side of OB. Now the circle whose centre is G and radius GD or GE, being orthogonal to the circles (O) and (C) and to OA, inverts them into themselves, and $\therefore$ inverts (P) into (C'), because both of these touch them;

$$\therefore\ GL\,.\,GD'=R^2\,;\ \therefore\ GL=\frac{R^2}{R+2r} \text{ and } OL=R+r-\frac{R^2}{R+2r}=\frac{r(3R+2r)}{R+2r}.$$

Further, GPC' is a st. line;

$$\therefore\ PL:C'D'=GL\,.\,GD'\,;\ \therefore\ PL=\frac{rR^2}{(R+2r)^2}\,;$$

$$\therefore\ PL:OL\ =R^2:(2r+3R)(2r+R),$$

$$i.e.\ PL:OL\ =1:(2\sqrt{2}+1)(2\sqrt{2}-1)=1:7.$$

Mr. Youngman notes as other properties of the figure:

1. The distances of N from OA and OB are as 1 to 4.
2. The points $ANFA'$ are concyclic; the centre is B' diametrically opposite to B in the circle (O).
3. DE and LM pass through B'.
4. $DLME$ are concyclic.

292. [D. 2. b. β.] *If* $f(x)=a_0+a_1x+a_2x^2+\ldots$ *be a series whose sum is known, show that the sum of every* n^{th} *term beginning with the* m^{th}, $(m \not> n)$ *is*

$$\frac{1}{n}\sum_{r=n}^{r=1}[f(\omega_r x)\times\omega_r^{\,n-m+1}],$$

where $\omega_1, \omega_2, \ldots \omega_n$ *are the* n^{th} *roots of unity. Apply this to sum the series*

$$(a)\ \frac{x^2}{2}\sin 2\theta+\frac{x^5}{5}\sin 5\theta+\ldots;\quad (b)\ 1-\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{9}-\ldots.$$

T. ROACH.

Solution by PROPOSER.

Let $a_0+a_1x+a_2x^2+\ldots=f(x)$ be a given series in powers of x of which the sum is known, to find the sum of every n^{th} term, beginning with the m^{th}, where m is not greater than n.

Let $a_1, a_2, \ldots$ be the n^{th} unit of unity, then if we multiply each term of the above series by $\omega_1^{\,n-m+1}$ and write $\omega_1 x$ for x, we have

$$a_0\omega_1^{\,n-m+1}+a_1\omega_1^{\,n-m+2}x+a_2\omega_1^{\,n-m+3}x^2+\ldots=f(\omega_1x)\times\omega_1^{\,n-m+1}.$$

Similarly, $a_0\omega_2^{\,n-m+1}+a_1\omega_2^{\,n-m+2}x+a_2\omega_2^{\,n-m+3}x^2+\ldots=f(\omega_2x)\times\omega_2^{\,n-m+1}$;

$$\therefore\ a_0\Sigma\omega^{n-m+1}+a_1x\Sigma\omega^{n-m+2}+a_2x^2\Sigma\omega^{n-m+3}+\ldots=\Sigma\{f(ax)\times\omega^{n-m+1}\}.$$

Now, all the summations on the left-hand side $=0$, except the n^{th} ones beginning with the m^{th}, and these each $=n$;

$$\therefore\ n(a_{m-1}x^{m-1}+a_{m+n-1}x^{m+n-1}+\ldots)=\Sigma\{f(ax)\times\omega^{n-m+1}\}.$$

Find the sum of every fourth term of the series

$$x\sin\theta+\frac{x^2}{2}\sin 2\theta+\frac{x^3}{3}\sin 3\theta+\ldots \text{ beginning with } \frac{x^2}{2}\sin 2\theta.$$

Here we have
$$K=\frac{1}{2i}\left\{xe^{i\theta}+x^2\frac{e^{2i\theta}}{2}+\ldots-xe^{-i\theta}-x^2\frac{e^{-2i\theta}}{2}-\ldots\right\}$$
$$=\frac{1}{2i}\{-\log(1-xe^{i\theta})+\log(1-xe^{-i\theta})\}$$
$$=\frac{1}{2i}\log\frac{1-x\cos\theta+x\,.\,i\sin\theta}{1-x\cos\theta-x\,.\,i\sin\theta}=\frac{1}{2i}\,.\,2i\tan^{-1}\frac{x\sin\theta}{1-x\cos\theta}.$$

$$\therefore\ S=\frac{1}{n}\Sigma\omega^{n-m+1}\tan^{-1}\frac{\omega x\sin\theta}{1-\omega x\cos\theta}.$$

Now, $n=4$, $m=3$, $\omega_1, \omega_2, \omega_3, \omega_4=1, -1, i, -i$.

$$\therefore\ S=\frac{1}{4}\left\{\tan^{-1}\frac{x\sin\theta}{1-x\cos\theta}+\tan^{-1}\frac{-x\sin\theta}{1+x\cos\theta}-\tan^{-1}\frac{ix\sin\theta}{1-ix\cos\theta}-\tan^{-1}\frac{-ix\sin\theta}{1+ix\cos\theta}\right\}$$
$$=\frac{1}{4}\left\{\tan^{-1}\left(\frac{x^2\sin 2\theta}{1-x^2\cos 2\theta}\right)-\tan^{-1}\left(\frac{-x^2\sin 2\theta}{1+x^2\cos 2\theta}\right)\right\}$$
$$=\frac{1}{4}\tan^{-1}\frac{2x^2\sin 2\theta}{1-x^4}.$$

Sum the series $1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{9}-\frac{1}{11}-\frac{1}{13}+\ldots$.

We have
$$x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\ldots=\tan^{-1}x.$$
$$\therefore\ x+\tfrac{1}{9}x^9+\ldots=\tfrac{1}{8}\Sigma\omega^7\tan^{-1}\omega x,$$
$$-\tfrac{1}{3}x^3-\tfrac{1}{11}x^{11}-\ldots=\tfrac{1}{8}\Sigma\omega^5\tan^{-1}\omega x,$$
$$\tfrac{1}{5}x^5+\tfrac{1}{13}x^{13}+\ldots=\tfrac{1}{8}\Sigma\omega^3\tan^{-1}\omega x,$$
$$-\tfrac{1}{7}x^7-\tfrac{1}{15}x^{15}-\ldots=\tfrac{1}{8}\Sigma\omega\tan^{-1}\omega x.$$
$$\therefore\ x-\tfrac{1}{3}x^3-\tfrac{1}{5}x^5+\tfrac{1}{7}x^7+\ldots=\tfrac{1}{8}\Sigma(\omega^7+\omega^5-\omega^3-\omega)\tan^{-1}\omega x$$
$$=\tfrac{1}{8}\Sigma(\omega^4-1)(\omega^3+\omega)\tan^{-1}\omega x.$$

Now, $\qquad \omega = \cos\frac{r\pi}{4} \pm i\sin\frac{r\pi}{4}$, where $r = 0, 1, 2, 3, 4$;

$\therefore\ \omega^4 - 1 = 0$ or -2, as r is even or odd;

$$\therefore\ S = -\frac{1}{4}\Sigma\left(\cos\frac{3r\pi}{4} \pm i\sin\frac{3r\pi}{4} + \cos\frac{r\pi}{4} \pm i\sin\frac{r\pi}{4}\right)\tan^{-1}\left(\cos\frac{r\pi}{4} \pm i\sin\frac{r\pi}{4}\right)x,$$

where $r = 1$ or 3

$$= -\frac{1}{4}\Sigma\, 2\cos\frac{r\pi}{4}\left(\cos\frac{2r\pi}{4} \pm i\sin\frac{2r\pi}{4}\right)\tan^{-1}\left(\cos\frac{r\pi}{4} \pm i\sin\frac{r\pi}{4}\right)x$$

$$= -\frac{1}{2}\Sigma\cos\frac{r\pi}{4}\, i\sin\frac{r\pi}{2}\tan^{-1}\frac{2i\sin\frac{r\pi}{4}x}{1+x^2}$$

$$= \frac{1}{4}\Sigma\cos\frac{r\pi}{4}\sin\frac{r\pi}{2}\log\frac{1+x^2+2x\sin\frac{r\pi}{4}}{1+x^2-2x\sin\frac{r\pi}{4}}$$

$$= \frac{1}{4}\left\{\frac{1}{\sqrt{2}}\log\frac{1+x^2+\sqrt{2}x}{1+x^2-\sqrt{2}x} + \frac{1}{\sqrt{2}}\log\frac{1+x^2+\sqrt{2}x}{1+x^2-\sqrt{2}x}\right\} = \frac{1}{2\sqrt{2}}\log\frac{1+x^2+\sqrt{2}x}{1+x^2-\sqrt{2}x}.$$

If $x = 1$, we have

$$1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} - \ldots = \frac{1}{2\sqrt{2}}\log\frac{2+\sqrt{2}}{2-\sqrt{2}} = \log(\sqrt{2}+1)^{\frac{1}{\sqrt{2}}}.$$

As an additional example, sum the series

$$2 - \frac{\sin^2 2\theta}{1-\cos 2\theta} + \frac{\sin^4 2\theta}{4(1-\cos 2\theta)^2} + \frac{\sin^6 2\theta}{12(1-\cos 2\theta)^3} + \frac{\sin^8 2\theta}{192(1-\cos 2\theta)^4} - \frac{\sin^{10} 2\theta}{1920(1-\cos 2\theta)^5}.$$

Here $\qquad S = 2 - 2\cos^2\theta + \dfrac{2^2\cos^4\theta}{2\,.\,2!} + \dfrac{2^3\cos^6\theta}{2^2\,.\,3!} + \ldots,$

and writing x for $\cos^2\theta$,

$$S = 2\left\{1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots\right\} = 2\left\{e^x - 2\left(x + \frac{x^5}{5!} + \ldots\right)\right\}.$$

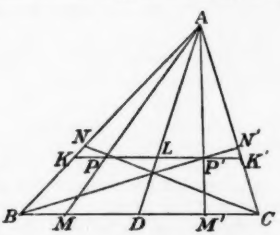
Now, $\qquad x + \dfrac{x^5}{5!} + \ldots = \dfrac{1}{4}(e^x - e^{-x} - ie^{ix} + ie^{-ix})$

(cf. Hayward's *Vector Alg.* p. 246);

$$\therefore\ S = 2e^x - \{e^x - e^{-x} - i(2i\sin x)\} = e^x + e^{-x} - 2\sin x = 2\{\cosh\cos^2\theta - \sin(\cos^2\theta)\}.$$

293. [J. 2. f.]. *If two random points are taken within a given triangle, show that their join is equally likely to cut any pair of sides of the triangle.* C. Sandberg.

Solution by C. Williams.



ABC the triangle, AD a median, P one of the points, PLP' parallel to BC,

$$PL = P'L.$$

Then if a second point Q be taken, PQ will cut AC if Q lies within APC or $BMPN$.

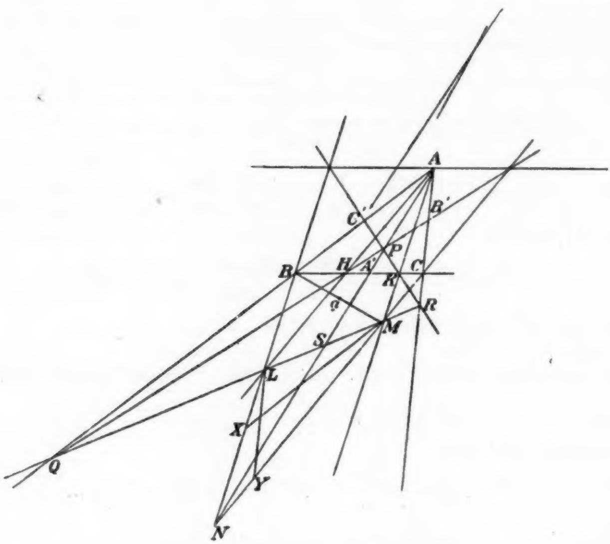
And if P' be the first point, $P'Q$ will cut AB if Q lies within $AP'B$ or $CM'P'N'$.

But it can easily be proved that $APC = AP'B$ and that $BMPN = CM'P'N'$. Therefore by taking pairs of points on KK' equidistant from L and the

second point Q anywhere, it follows that the chance of the join cutting AB is equal to the chance of its cutting AC. And the same will be true by taking the first point on any other line parallel to BC, and therefore it is true generally.

294. [K. 1. c.]. *Any two lines l, m, are drawn through the vertex A of the triangle ABC. BL parallel to m meets l in L; CM parallel to l meets m in M. BL cuts CM in N. A', B', C' are mid points of sides. Prove $A'N$, $B'M$, $C'L$ are concurrent.*

J. A. Third.



Solution by H. G. Mayo, H. A. Saunders, J. C. Palmer.

Complete the parallelograms whose sides are parallel to l, m, and whose diagonals are the sides of the triangle. Then $A'N$, $B'M$, $C'L$ being produced are the concurrent diagonals of these parallelograms.

Mr. Saunders notes the following extension:

If AL and AM cut BC in H and K, α and β are the middle points of BM and CL, and parallelograms $ABXM$, $ACYL$ are completed, then the following sets of lines are concurrent:

(i.) AC, $C'K$, LM, (iii.) AN, $B'H$, $C'K$,

(ii.) AB, $B'H$, ML, (iv.) LC', $N\alpha$, YA',

(v.) MB', $N\beta$, XA'.

Also the intersections of MC and HB', and of LB and KC', lie on a parallel to BC through A.

For parallels to AM and AL through C' and B' respectively both bisect LM. $B'C'S$ and BHL are $\triangle$s with their sides parallel. Hence AB, $B'H$, and ML are concurrent. Similarly for (i.).

Also AHK, $B'C'S$ are $\triangle$s with their sides parallel;

$\therefore$ AS, $B'H$, $C'K$ concur. And since S is the middle point of LM, AS passes through N.

Similarly $C'A'a$, LYN have their sides parallel, which gives (iv.). Similarly (v.).

Also the points of concurrence of (i.) and (ii.) being R and Q respectively, MKC, $QC'B'$ are in perspective;

$\therefore$ MC and HB' intersect on the line joining A to the intersection of KC, $B'C'$, etc.

MR. YOUNGMAN remarks that this is another statement of the theorem given in Smith's *Conic Sections* (chap. ii.). If on the sides of a triangle as diagonals parallelograms are described having their sides parallel to two given straight lines, the other diagonals will concur.

When the parallelograms are rectangles, any two of these diagonals meet on the circle $A'B'C'$, and hence all three concur; and by orthogonal projection it follows that the theorem is true for any parallelograms.

Statical considerations give an easy proof, thus: forces AL, AM, NL, NM are in equilibrium; but taken by pieces (NB, NC), (CM, AM), and (AL, BL) they compound into $2NA'$, $2B'M$, $2C'L$; therefore $A'N$, $B'M$, $C'L$ are concurrent.

It may be noticed that $B'L$ and $C'M$ meet on the parallel through A to $A'N$. The forces LB', MC' are equivalent to LA, AB', MA, AC', and therefore to NA, AA'; so that their resultant is equal and parallel to NA'; and it acts through A, because the areas ALB', AMC' are equal, each being a quarter of the parallelogram $ALNM$.

Other forms of the theorem are: Question 186 (see p. 140), a proof of which is given in Casey's *Elements*, after I. 43; and Wolstenholme's *Problems*, 72, 73.

A general enunciation of Question 186 would run: Through any point in a parallelogram lines are drawn across it parallel to its sides, so that there are altogether nine parallelograms in the figure. If any three of these be so chosen as to have for diagonals the sides of a triangle, their other diagonals will meet in a point. Six such points can be obtained, lying three and three on two straight lines.

295. [J. 2. f.]. *Prove that Christmas Day is least likely to fall on Monday or Saturday, and most likely on Sunday, Tuesday, or Friday; and find numbers proportional to the probabilities of these events.* (Welsh Medal, Kingswood School.) T. P. THOMPSON.

Solutions by E. H. DOUGLAS, R. P. KNAPTON, C. E. YOUNGMAN.

Since after every 400 years the calendar begins to repeat itself exactly, it will be enough to consider the period 1898-2297, with the knowledge that in 1898 Christmas was on Sunday. In every ordinary year Christmas falls one day later in the week, and in leap-year two days later, than in the year before; that is, it goes the round of the week, missing a day in leap-year. Now in 400 years there are 97 leap-years; therefore $497 \div 7$, or 71, will be the number of these rounds. Each day is missed once in 28 years, in the order, Sat. Th. Tu. Sun. F. W. M., so long as leap-year does not fail. The first to be missed is Sat. in 1904; add 7×28, and we come to 2100, which therefore, not being leap-year, keeps Christmas on Sat. The next day missed is W. in 2104; add 3×28; $\therefore$ W. is missed in 2188, M. in 2192, Sat. in 2196, and Christmas in 2200 falls on Th. Hence M. is missed in 2204; add 3×28; $\therefore$ again M. in 2288, Sat. in 2292, and finally Th. in 2296. Thus each day is missed 13 times, and in addition W. and Th. once, and M. and Sat. twice. Subtracting from 71, we find that in every 400 years M. and Sat. have Christmas 56 times, and Sun., Tu., and F. 58 times each. Therefore the numbers required are 112 and 174, or 56 and 87.

296. [R. 1. d.] *BC, AC are straight paths. A′, B′, two men, start from A and B and run with uniform speed towards C. As they run they come into a straight line with a post P, and again with a post Q; in each case the post is half-way between them. When B′ has reached C they stop running, and walk at quick march to meet each other. Show (geometrically) that, when they meet, the posts P and Q will be in a straight line with them.* R. B. WORTHINGTON.

Solution by C. E. YOUNGMAN.

Because of the uniform speed, the ratio $AA' : BB'$ is constant. Therefore by Newton's lemma (*Principia*, I. 23) the mid point of $A'B'$ describes a straight line, which must be PQ; $\therefore$ PQ cuts AC halfway between the men's positions when they stop running. But this is just where they meet, since they both walk at the quick march.

Solution by C. WILLIAMS.

If A' be reduced to rest, and the direction of relative motion of B' be BK, cutting AC at K, then it is evident that if A be joined to any number of positions along BK, the mid-points of these joins are collinear.

Solution by R. P. KNAPTON, J. C. PALMER, A. POOLE.

(Fig. 1). When A is at A_1, A_2, C,

B is at B_1, B_2, C_3 respectively,

where $A_1A_2 : B_1B_2 = A_2C : B_2C_3$.

Let $CC_3 = x$, and move B_1, B_2, C_3 along B_1C a distance x. Then B_1, B_2, C_3 come into positions C_1, C_2, C.

Also $A_1A_2 : B_1B_2 = A_2C : B_2C_3$;

$\therefore$ $A_1A_2 : C_1C_2 = A_2C : C_2C$;

$\therefore$ A_1C_1, A_2C_2 are $\|^{l}$.

Hence if L, M are mid-pts. of A_1C_1, A_2C_2, we have LP and MQ are $\|^{l}$ to B_1C, and each $=\frac{1}{2}x$; $\therefore$ the mid-pt. of CC_3, Q and P are in one st. line.

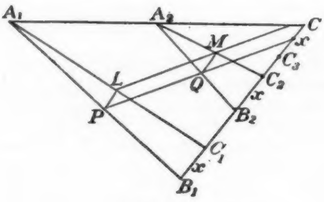


FIG. 1.

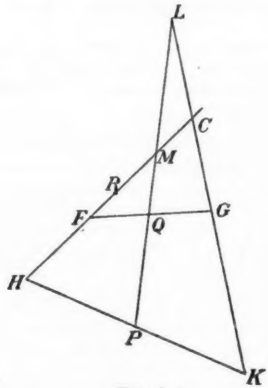


FIG. 2.

Solution by H. A. SAUNDERS.

(Fig. 2). Let H, K be the positions of A' and B' when they are in a line with P and F, G when in a line with Q.

Suppose A' is at R when B' is at C.

Let PQ produced cut HC in M and KC produced in L. Then LQP being a transversal and $HP=PK$,

$$KL \cdot CM = LC \cdot HM. \quad \text{.........(i.)}$$

Similarly
$$GL \cdot CM = CL \cdot FM; \quad \text{.........(ii.)}$$

$$\therefore GK \cdot CM = HF \cdot CL; \quad \text{.........(iii.)}$$

$$\therefore \frac{KL}{GL} = \frac{HM}{FM}; \quad \text{.........(i.)} \div \text{(ii.)}$$

$$\therefore \frac{GK}{GL} = \frac{HF}{FM}; \quad \therefore \frac{GL}{FM} = \frac{GK}{HF} = \frac{CG}{FR} = \frac{GL-GC}{FM-FR} = \frac{CL}{RM}.$$

But
$$\frac{GK}{HF} = \frac{CL}{CM};$$

$$\therefore CM = RM,$$

and if they walk at the same rate they meet at M.

297. [L^1. 10. a. b.; K. 16. f.]. *Find the locus of the centre of a sphere which rolls on a parabolic wire, touching it at two points.* (C.)

Solution by C. Williams.

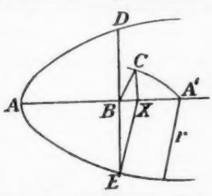
If the parabola be $z^2=4ax$, and if C be the centre of the sphere of radius r, P a point of contact, then, with usual lettering,

$$r^2 = CP^2 = CG^2 + NG^2 + PN^2 = y^2 + 4a^2 + 4ax,$$

which is an equal parabola.

Solution by W. S. Cooney.

From the symmetry of parabola and sphere, locus of centre is evidently in a plane perpendicular to axis and plane of parabola, intersecting the axis at A', where the normal to parabola in its plane equals radius (r) of sphere.



Let C be a position of centre, D and E points of contact with wire, and CX distance of C from plane. C, E, and X lie in a plane normal to parabolic plane and curve at E, EX being the normal and BX the subnormal. As centre of sphere may be supposed for a small movement to turn round DE, CB must be normal to curve $A'C$, and BX its subnormal, $\therefore$ locus of C is a parabola equal to parabolic wire (having the same constant subnormal) and independent of r, except for the position of its vertex A'.

298. [K. 20. c. a.]. *Find the sum and product of the non-zero roots of* $\Sigma \tan^{-1}(a-x) = \Sigma \tan^{-1} a = \tan^{-1} s.$ (C.)

Solution by C. Williams.

Let
$$p_2 = ab + ac + bc, \quad p_3 = abc,$$

$$\Sigma \tan^{-1} a = \tan^{-1} s, \quad \therefore \frac{2s - p_3}{1 - p_2} = s, \quad \therefore p_3 = s(1+p_2);$$

also
$$\Sigma \tan^{-1}(a-x) = \tan^{-1} s, \quad \therefore \frac{\Sigma(a-x) - (a-x)(b-x)(c-x)}{1 - \Sigma(a-x)(b-x)} = s;$$

$$\therefore \frac{2s - 3x - p_3 + p_2 x - 2sx^2 + x^3}{1 - p_2 + 4sx - 3x^2} = s,$$

$$\therefore x^3 + x^2 \cdot s + x(p - 3 - 4s^2) = 0,$$

$$\therefore x_1 + x_2 = -s, \quad x_1 x_2 = p_2 - 3 - 4s^2.$$

300. [K. 1. c; K. 4.] *P and Q are the centres of the squares inscribed in a right-angled triangle.*

(a) *Prove the projections of PQ on the three sides are concyclic.*

(β) *Given PQ and side of either inscribed square construct triangle.*

W. S. Cooney.

Solution by Proposer and C. E. Youngman.

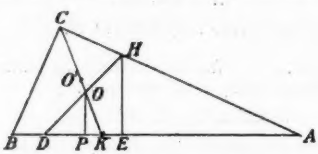
(a) Let O, O' be centres of insquares, $\therefore$

$$DE=s;\ \cot\angle OAB=\frac{AE+EP}{OP}=\frac{s\,.\cot A+\frac{1}{2}s}{\frac{1}{2}s}=1+2\cot A.$$

Similarly, $\cot\angle O'AC=1+2\cot A$; $\therefore$ $\angle OAB=\angle O'AC$.

Similarly, $\angle OBA=\angle O'BC$ and $\angle OCB=\angle O'CA$, but $O'C$ bisects $\angle C$;

$\therefore$ O, O', and C are collinear, and O, O' are isogonal conjugates with respect to $\triangle ABC$; $\therefore$ their projections on the three sides are concyclic.



(β) C, H, K, D are concyclic since $\angle ODK=\angle OCH=45°$;

$\therefore$ rect. $CO\,.\,OK=OD^2$; $\therefore$ $O'K^2=OO'^2+OD^2$;

$\therefore$ if OD be given, $O'K$ and OK are known, and $\angle ODK$ being 45°, $\triangle ODK$ can be constructed and then $\triangle ABC$.

If $O'K$ be given, then OK and OD are known; $\therefore$ etc.

301. [L¹. 4. c; 7. d.] *Find (geometrically) the locus of the foci of all parabolas touching two fixed straight lines.* C. V. Durell.

Solution by R. B. Worthington.

Let S be a focus of a conic which touches TP and TQ at P, Q. Required the locus of S.

Construction. Draw the straight lines SP, SQ, SHT, $QHPK$.

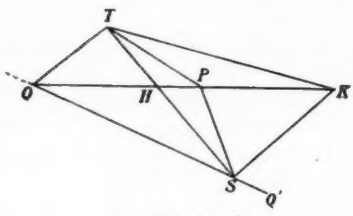


Fig. 1.

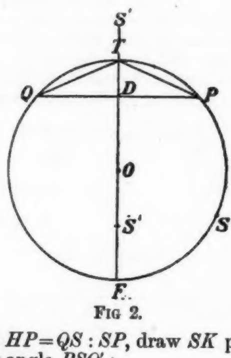


Fig 2.

Analysis. $\therefore$ ST bisects $\angle PSQ$; $\therefore$ $QH:HP=QS:SP$, draw SK perpendicular to ST; then SK bisects the exterior angle PSQ';

$\therefore$ $QK:KP=QS:SP=QH:HP$;

$\therefore$ $QHPK$ is an Harmonic range.

SYNTHESIS. Let TS be any straight line through T.

Draw TK a fourth harmonic to TQ, TH, TP, and draw KS perpendicular to TH; then S is a focus of a conic touching TP, TQ in P, Q. Proof obvious.

Hence we know the intersection of the locus of S with all straight lines through T. The locus of S therefore is geometrically determined.

Particular case when $TP=TQ$.

About $\triangle TPQ$ describe a circle $PEQT$. Clearly, if S is upon the arc PEQ, or on the diameter TE (or this produced), and nowhere else, the lines TP, TQ subtend equal angles at S.

The circle therefore and its diameter combined constitute the locus of S.

The touching conic is throughout considered as an ellipse, but (*mutatis mutandis*) the construction includes all conics.

The particular case shows that a straight line may cut the curve in three points. The locus is therefore not a conic.

Solution by E. P. ROUSE.

If it is sufficient to find geometrically any number of points on the locus, the following construction may serve. It is based on a Proposition proved in No. 4 of *The Mathematical Gazette.*

PQ, PQ' being the given tangents at Q and Q', bisect QQ', draw the median line PC, draw also the symmedian line cutting the circle PQQ in U. In PC take any point C; bisect the angle PCU by a straight line, along which mark off $CS=CS'=\sqrt{CP.CU}$; then S, S' are foci.

If we take P for the origin of rect. coord., PC the axis of x, and make a, $-2b$ the coordinates of U, the equation to the locus becomes

$$y(x^2+y^2)-b(x^2-y^2)-axy=0,$$

on rejecting the solution $y+b=0$, which corresponds to the case when the conic is a parabola, for in that case the focus bisects PU, whatever value a may have.

Problem 262 in No. 17 is a special case of the Proposition referred to, viz. when P is a point on the curve.

Solution by C. E. YOUNGMAN.

Let OA, OB (fig. 3) be the fixed lines, A and B the fixed points. Draw a circle with centre O, and from A and B the tangents APQ, $AP'Q'$, BPQ', $BP'Q$. It is readily proved that PP', QQ' are pairs of foci, and that the locus might be defined as that of a point P such that OP is one of the bisectors of angle APB (which entails that any line through O can only cut the locus once); also that OP, OP' are equally inclined to OA, OB; and that C, the centre of the conic describes the straight line OC which bisects AB. Find F, the finite focus of the parabola which touches OP, OP' at P and P' (the infinite focus lies on OC); $\therefore$ OF, OC are equally inclined to OP, OP', and consequently to OA, OB); draw FT, FT' tangents to the circle O. Since this circle is circumscribed by the quadrilateral $PQP'Q'$, the pencil $F(TT', PP', QQ', AB)$ is in involution. But the angles of the first two pairs of rays have the same bisectors, FO and its perpendicular; $\therefore$ also the angles of

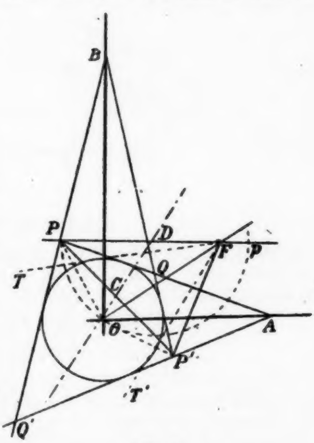


FIG. 3.

the second two, *i.e.* FO bisects angle AFB; $\therefore$ F is a point on the locus. But OF is a fixed line; $\therefore$ F is a fixed point.

Let FP meet OC at D. From similar triangles FPO, FOP' we have $\angle DPO = FOP' = DOP$; $\therefore$ $DP = DO$. Thus points on the locus may be obtained by drawing any line FD cutting OC at D, and taking on it in either direction $DP = Dp = DO$; $\therefore$ the locus is a strophoid with focus F and directrix OC; node at O, where the tangents bisect the angle AOB.

This curve is discussed by Jeřabek in Mathesis, 1896, p. 38, in connection with the parabola enveloped by PP'.

302. [K. 1. b. α.] *PL, PM, PN are perpendiculars on the sides of a triangle ABC from any point P on the bisector of A. MN meets BC in X. If D be mid-point of BC, show that $DL : DX = (b-c)^2 : a^2$.* R. W. GENESE.

Solution by W. S. COONEY.

(Fig. 1) On AC make $AQ = AB$; join BQ meeting bisector of $\angle A$ in E, then $BE = EQ$, and let DE, which is parallel to CA, meet MN in H. MN is parallel to BQ.

$$P\hat{E}B = P\hat{L}B = P\hat{M}B = 90^\circ; \quad \therefore P, E, L, B, M \text{ are concyclic.}$$

But $$\angle EHM = \angle HEB = \angle MBE; \quad \therefore HMBE \text{ is cyclic.}$$

$$\therefore \frac{DL}{DH} = \frac{DE}{DB} = \frac{CQ}{CB} = \frac{b-c}{a} = \frac{CN}{CX} = \frac{DH}{DX};$$

$$\therefore \frac{DL}{DX} = \frac{(b-c)^2}{a^2}.$$

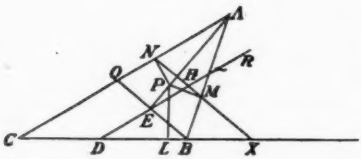


FIG. 1.

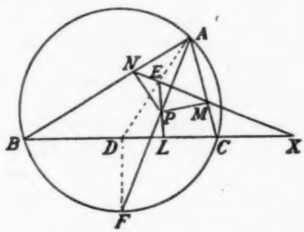


FIG. 2.

Solution by C. E. YOUNGMAN.

(Fig. 2) Let PL meet MN at E, and AP meet circle ABC at F, D being the projection of F on BC. Then, wherever P is, the triangles EPM, BFA are equiangular, and also MAP, DBF;

$$\therefore EP : PM = BF : FA \text{ and } PM : PA = FD : FB;$$

$$\therefore EP : PA = DF : FA; \quad \therefore AED \text{ is a straight line.}$$

Also angle $AMN = APN = 90^\circ - NAP$; $\therefore$ as P moves along AF, E moves along AD, while EL and MN move parallel to themselves. $\therefore$ figure $EDLX$ remains similar to itself, and the ratio $DL : DX$ is constant.

When APF is the bisector of A, D is mid-point of BC, and putting P at A we have $DL = (b^2 - c^2)/2a$, and $BX : CX = c : b$; whence DX can be found, and the ratio as given.

In general, when $\sin PAB = k \sin PAC$,

$$DL : DX = (\cos B - k \cos C)(k \cos B - \cos C) : (1 + k \cos A)(k + \cos A).$$

304. [A. 1. c. β.] *If a, b, c, d are integers, and b, d contains no square factors, under what conditions will $\sqrt{a\sqrt{b}+c\sqrt{d}}+\sqrt{a\sqrt{b}-c\sqrt{d}}$ be the fourth root of an integer?* A. LODGE.

Solution by J. L. THOMAS.

If $$\sqrt{a\sqrt{b}+c\sqrt{d}}+\sqrt{a\sqrt{b}-c\sqrt{d}}=\sqrt[4]{p},$$
then $$2a\sqrt{b}+2\sqrt{a^2b-c^2d}=\sqrt{p},$$
and $$8a^2b-4c^2d+8a\sqrt{a^2b^2-c^2bd}=p,$$
$\therefore$ since p is an integer, $a^2b^2-c^2bd$ is a perfect square;
$\therefore$ $c^2d=mb$, where m is integral;
$\therefore$ a^2-m is a perfect square, $=\kappa^2$ say.

Then
$$\begin{aligned} p=8a^2b&=4mb+8ab\sqrt{a^2-m}\\ &=4b(2a^2-m+2a\sqrt{a^2-m})\\ &=4b(a+\sqrt{a^2-m})^2\\ &=4b(a+\kappa)^2 \end{aligned}$$
$$\therefore\ \sqrt[4]{p}=\{4b(a+\kappa)^2\}^{\frac{1}{4}}.$$

Thus in 151, p. 87, No. 10,
$$\sqrt[4]{p}=\sqrt{9\sqrt{6}+6\sqrt{12}}+\sqrt{9\sqrt{6}-6\sqrt{12}},$$
$$a=9,\ b=6,\ c^2d=36\times 12=6\times 72=6m,\ a^2-m=81-72=9=\kappa^2;$$
$$\therefore\ \sqrt[4]{p}=\sqrt[4]{4\cdot 6(9+3)^2}=\sqrt[4]{24\cdot 144}=2\sqrt[4]{216}.$$

305. *If two identically equal polygons $ABC\ldots A'B'C'\ldots$ are situated anywhere in the same plane, prove that either the perpendicular bisectors of AA', BB', $CC'\ldots$ are concurrent, or the mid-points of AA', BB', $CC'\ldots$ are collinear. What are the corresponding theorems for any two similar polygons situated in the same plane?* F. S. MACAULAY.

Solution by W. S. COONEY.

(1) If the polygons are equal and directly placed (fig. 1), let OP and OP' be perpendiculars from middle points of AA' and BB', then
$$AO=A'O,\ BO=B'O,\ AB=A'B'. \quad \therefore\ \angle ABO=\angle A'B'O.$$
But $\angle ABC=\angle A'B'C'$; $\therefore$ $\angle OBC=\angle OB'C'$; $\therefore$ $OC=OC'$.
Similarly, $OD=OD'$; $\therefore$ etc.

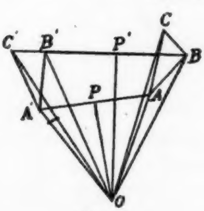


FIG. 1.

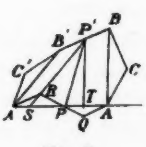


FIG. 2.

Now if ABC, etc., be directly similar to $A'B'C'$, etc., and O be taken so that $AO:A'O=BO:B'O=$ common ratio of sides of figures, then evidently $\triangle AOB$ is similar to $\triangle A'OB'$, and then $\triangle OBC$ similar to $\triangle OB'C'$; $\therefore$ $OC:OC'=$ common ratio, and similarly for $OD:OD'$. $\therefore$ corresponding theorem is that the loci of vertices of triangles on the bases AA', BB', CC', etc., having the same ratio of sides, have a common point.

(2) If polygons are inversely placed complete the parallelograms $ABP'Q$ and $A'B'PR$ (fig. 2), join PQ and PR, then evidently $\triangle APQ=\triangle A'PR$;

$\therefore$ QPR is a straight line, $\triangle PP'Q = \triangle PP'R$, and PP' bisects angle $QP'R$;

$$\therefore \angle P'PA = \tfrac{1}{2}(\angle P'ST + \angle P'TA) = \tfrac{1}{2}(\angle B'A'A + \angle BAW).$$

Now if P″ be middle point of CC', then in the same way

$$\angle P''PA = \tfrac{1}{2}(\angle C'A'A + \angle CAW) = \tfrac{1}{2}(\angle B'A'A + \angle BAW),$$

since $\angle B'A'C' = \angle BAC$; $\therefore$ P'', P', P are collinear; $\therefore$ etc.

If figures be inversely similar, and P and P' divide AA' and BB' in the common ratio, then

$$AQ : A'R = AP : A'P; \quad \therefore \quad PQ : PR = \text{common ratio} = P'Q : P'R;$$

$\therefore$ again $P'P$ bisects angle $QP'R$; $\therefore$ corresponding theorem is that the points dividing AA', BB', CC', etc., in the common ratio are collinear.

309. [K. 5. a; 23.] *Similar isosceles triangles are described on the sides of a triangle ABC. Show that the triangle formed by their vertices is in perspective with ABC, and find the locus of the centre of perspective, and the equation to the axis of perspective.* (Trin. C., 1897.) T. S. Munster.

Solution by J. F. Hudson.

Let $A'B'C'$ be the vertices of the similar isosceles triangles, and the angle $BAC' = \theta$. Then the trilinear coordinates of C' are $\frac{c}{2}\sec\theta \cdot \sin(B+\theta)$, $\frac{c}{2}\sec\theta \cdot \sin(A+\theta)$, $\frac{c}{2}\tan\theta$, and the equation of CC' is

$$\alpha\sin(A+\theta) - \beta\sin(B+\theta) = 0.$$

$\therefore$ the coordinates of the centre of perspective are given by

$$\alpha\sin(A+\theta) = \beta\sin(B+\theta) = \gamma\sin(C+\theta). \quad \text{.....................(i.)}$$

Eliminating θ from (i.), we find that the locus of the centre of perspective is

$$\alpha\beta\sin(A-B) + \beta\gamma\sin(B-C) + \gamma\alpha\sin(C-A) = 0,$$

a rectangular hyperbola, passing through the vertices, orthocentre, and centre of gravity of the triangle ABC.

The equation of $C'A'$ is

$$\begin{vmatrix} \alpha, & \beta, & \gamma, \\ \sin(B+\theta), & \sin(A+\theta), & \sin\theta, \\ \sin\theta, & \sin(C+\theta), & \sin(B+\theta) \end{vmatrix} = 0.$$

This meets CA where

$$\alpha(\sin\overline{A+\theta}.\sin\overline{B+\theta} - \sin\theta.\sin\overline{C+\theta}) = -\gamma(\sin\overline{B+\theta}.\sin\overline{C+\theta} - \sin\theta.\sin\overline{A+\theta}),$$

or $$\alpha(\cos A\cos B + \cos 2\theta\cos C) + \gamma(\cos B\cos C + \cos 2\theta\cos A) = 0;$$

$\therefore$ the equation of the axis of perspective is

$$\alpha/(\cos B\cos C + \cos A\cos 2\theta) + \beta/(\cos A\cos C + \cos B\cos 2\theta) + \gamma/(\cos A\cos B + \cos 2\theta\cos C) = 0.$$

Geometrical Proof that the triangles are in perspective.

Let AA', BB', CC' meet BC, CA, AB in P, Q, R respectively.

Then $\dfrac{AQ}{CQ} = \dfrac{\triangle BAB'}{\triangle CBB'}$, $\dfrac{CP}{BP} = \dfrac{\triangle ACA'}{\triangle ABA'}$, $\dfrac{BR}{AR} = \dfrac{\triangle CBC'}{\triangle ACC'}$;

$$\therefore \frac{AQ.CP.BR}{CQ.BP.AR} = \frac{BAB'.ACA'.CBC'}{CBB'.ABA'.ACC'} = -1.$$

Since $\triangle BCB' = \triangle ACA'$,

$\triangle ABA' = \triangle CBC'$,

$\triangle ACC' = \triangle BAB'$ (Euclid vi. 15);

$\therefore$ the triangles are in perspective.

314. [K. 2. e.] *ABC is a triangle. AL any line meeting BC in L, on it AP is taken so that $AP.AL = K^2$. From any point D on AB, DE is drawn parallel to AL to meet BC in E, and DF is taken so that $DF.DE = K^2$. BF produced cuts AL in G. Through P, G any circle is drawn; prove that the tangent AT to this circle is equal to DF.* R. TUCKER.

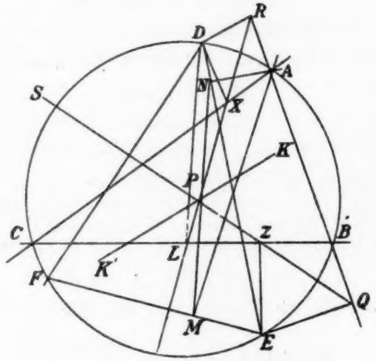
Solution by E. P. BARRETT; A. E. LAYNG; J. C. PALMER; J. L. THOMAS.

$$AT^2 = AP.AG;\ K^2 = AP.AL = DF.DE;$$

$$\therefore \frac{AT}{DF.DE} = \frac{AG}{AL} = \frac{DF}{DE} \text{ by parallels};$$

$$\therefore AT = DF.$$

315. [K. 1. c.] *If the pedal lines of D, E, F with respect to a triangle ABC meet in a point P, then* (1) *these pedals are perpendicular to EF, FD, DE;* (2) *P bisects the orthocentre-join of ABC and DEF;* (3) *the pedal lines of A, B, C for the triangle DEF pass through P.* C. F. YOUNGMAN.



Solution by W. S. COONEY.

(1) and (2). K and K' are orthocentres of ABC and DEF. The pedals of D and E intersect at an angle = to angle of segment DE, for

$$\angle RPS = \angle PRQ + \angle PQR = \angle ADX + \angle BEZ$$

$$= 90° - \angle \text{ on arc } DC + 90° - \angle \text{ on arc } CE = 180° - \angle \text{ on arc } DE.$$

If DF and EF be drawn perpendicular to PQ and PR, $\angle RPS = \angle F$;

$$\therefore F \text{ is on circle.}$$

DK' and RP are perpendicular to FE, and RP bisects DK;

$$\therefore RP \text{ bisects } KK'.$$

Similarly QP bisects KK'. PR and the pedal of F intersect at an angle $= D\hat{E}F$,

and as PR is perpendicular to FE, the pedal of F must be perpendicular to DE and $\therefore$ as before bisects KK';

$$\therefore \text{ (1) and (2) are proved.}$$

(3) Let MN be pedal of A to DEF, then $\angle DLX = \angle DCX$ ($DXLC$ is cyclic)

$$= \angle DEA = \angle NMA \text{ (}ANME\text{ is cyclic),}$$

but LX is parallel to MA;

$$\therefore \; DL \text{ is parallel to } NM;$$

$$\therefore \; NM \text{ is perpendicular to } BC,$$

and since NM bisects AK' it also bisects KK'. Similarly for the other pedals.

$$\therefore \text{ etc.}$$

316. [A. 1. b.] *If x, y, z, u are four positive quantities,*

$$3\Sigma(y+z+u)^{-1} \nless 16(x+y+z+u)^{-1},$$ E. M. Langley.

and the minimum value of

$$\Pi(x+1) \textit{ is } \frac{256}{81} \textit{ if } \Sigma\frac{u}{u+1}=1.$$ (Queen's, 1898, C.)

If a, b, c, d, e are all positive, $\Sigma\left(\frac{abc}{de}\right)^4 > \Sigma ab^2c.$ (St. Cath.'s, 1896.)

Solution by R. F. Davis.

If α, β, γ, δ, ... are n positive quantities,

$$\Sigma(\alpha).\Sigma(\alpha^{-1}) = 1+1+1\ldots n \text{ terms} + (\alpha/\beta+\beta/\alpha)+\ldots \frac{n(n-1)}{1.2} \text{ terms}$$

$$\nless n+n(n-1)$$

$$\nless n^2.$$

Put $\alpha=y+z+u$, $\beta=x+z+u$, $\gamma=x+y+u$, $\delta=x+y+z$, so that $\Sigma(\alpha)=3\Sigma(x)$.

Then substituting, we have the first inequality.

Next put $\alpha=1+1/x$, etc.; then by hypothesis $\Sigma(\alpha^{-1})=1$; $\therefore$ $\Sigma(\alpha)\nless 16$.

Therefore $\Sigma(1/x)\nless 12$; and using the theorem that the geometrical mean of any number of positive quantities is less than the arithmetical mean, we have

$$1/xyzu \ngtr 3^4,$$

and
$$xyzu/\Pi(x+1) \ngtr \left(\frac{1}{4}\right)^4, \text{ for } \Sigma\{x/(x+1)\}=1;$$

$$\therefore \; \Pi(x+1) \nless \left(\frac{4}{3}\right)^4.$$

In the last case Σ clearly cannot have its usual meaning, but is restricted by an implied reference to the cyclical order of a, b, c, d, e.

$$2\{ABC/DE+BCD/EA+\ldots+\ldots+EAB/CD\}$$

$$=A \; BC/DE+DE/BC\}+\ldots$$

$$\nless 2\{A+B+\ldots+\ldots+E\}.$$

Put $A=a^4$, $B=b^4$, etc., then

$$\Sigma\left(\frac{abc}{de}\right)^4 \nless \Sigma(a^4), \textit{ a fortiori } \nless \Sigma(a^2b^2), \textit{ a fortiori } \nless \Sigma(ab^2c),$$

for
$$(a^2-b^2)^2+(b^2-c^2)+\ldots+\ldots+(e^2-a^2)^2>0,$$

$$(ab-bc)^2+ \ldots\ldots\ldots\ldots\ldots\ldots +(ea-ab)^2>0.$$

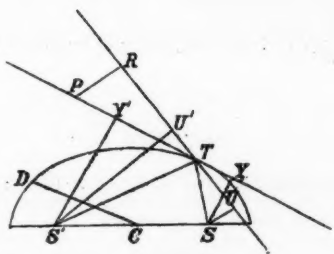
Mr. J. L. Thomas solves part (2) as follows:

$$\Sigma\frac{u}{u+1}=1. \quad \text{But } \Sigma\frac{u+1}{u+1}=4. \quad \therefore \ \Sigma\frac{1}{u+1}=3.$$

And $\Pi(u+1)$ is a min. when $\Pi\frac{1}{u+1}$ is a max., *i.e.* when $\frac{1}{u+1}=\frac{1}{z+1}=\ldots=\frac{3}{4}$.

$$\therefore \text{ the min. value of } \Pi(u+1) \text{ is } \left(\frac{4}{3}\right)^4=\frac{256}{81}.$$

318. [L^1. 17. e.] *From any point P on the outer of two similar coaxial ellipses, tangents PT, PT' are drawn to the inner, and a circle of given radius is described with the centre P. Find the envelope of the four tangents from T, T' to the circle.* (Gonv. and Caius, 1894.)



Solution by R. F. Davis.

Let TR be a tangent from T to the circle, centre P, whose radius PR = given constant; SU, $S'U'$ the perpendiculars upon it from the foci of the inner ellipse.

Then it may be shown by orthogonal projection or otherwise that

$PT=\lambda \, . \, CD$ (semi-conjugate diameter of inner ellipse to CT).

Hence $$SU \, . \, S'U'=ST \, . \, S'T\sin(S\hat{T}Y-U\hat{T}Y)\sin(S\hat{T}Y+UTY)$$
$$=ST \, . \, S'T(\sin^2 S\hat{T}Y-\sin^2 UTY)$$
$$=SY \, . \, S'Y'-PR^2/\lambda^2$$
$$=\text{constant}.$$

Thus the envelope of TR is a confocal ellipse.

319. [R. 1. d.] (*b*) *A spherical shell is dropping vertically, and just as it reaches the ground with a velocity V, it explodes so that each fragment of surface is driven out with a relative normal velocity u. Prove that if $u>V$ the fragments will be distributed on the ground within a circle of radius*

$$\frac{1}{8g}[\sqrt{V^2+8u^2})-3V][8u^2-2V^2-2V\sqrt{V^2+8u^2}]^{\frac{1}{2}}$$

where the radius of the shell is neglected in comparison with this length. (Trinity C., 1896.)

Solution by R. F. Davis, E. Ll. Tanner, J. L. Thomas.

Suppose a particle of the shell to receive a velocity u at an angle θ to the horizon.

Then it is projected with horizontal velocity $u\cos\theta$, and vertical velocity $u\sin\theta - V$, so that the range is

$$\frac{2u\cos\theta(u\sin\theta - V)}{g}.$$

This is a maximum when $(1-2\sin^2\theta)u + V\sin\theta = 0$,

$$\therefore\ 2u\sin^2\theta - V\sin\theta - u = 0,$$

$$\sin\theta = \frac{V \pm \sqrt{V^2+8u^2}}{4u}.$$

the range is

$$\frac{2u}{g}\,\frac{(\sqrt{V^2+8u^2} - 3V}{4}\cdot\frac{(16u^2 - 2V^2 - 8u^2 - 2V\sqrt{V^2+8u^2})^{\frac{1}{2}}}{4u}$$

since the negative sign is inadmissible

$$= \frac{1}{8g}(\sqrt{V^2+8u^2} - 3V)(8u^2 - 2V^2 - 2V\sqrt{V^2+8u^2})^{\frac{1}{2}}.$$

REVIEWS.

Sur les lois de Réciprocité. Par X. Stouff. (Paris: Hermann, 1898, 31 pp.) This is an enlarged edition of a note that appeared under the same title in the *Comptes rendus* of 1896. M. Stouff applies geometry of n dimensions (as developed in Minkowski's *Geometrie der Zahlen*) and certain groups of points in hyper-spaces to the discussion of Kummer's complex numbers and their power residues. The main interest of the paper is in the method, which is suggestive and has the charm that distinguishes a geometrical treatment of the theory of numbers. T.

Conférences sur les Mathématiques faites . . . à l'occasion de l'exposition de Chicago, par Félix Klein, recueillies par A. Ziwet, traduites par L. Laugel. (Paris: Hermann, 1898, 125 pp.) The Chicago *colloquia* are too well known to require a review; they are "maps of the country," made by a pioneer, with the exceptional quality of being not only useful to pioneers but also attractive to the tenderfoot. The translation before us will be welcome, partly because, as we are informed, the American edition is out of print; and even more because of the excellent bibliographical notices which have been added by M. Laugel, who has had the advantage of receiving some references from Professor Klein himself. T.

Sur les fondements de la théorie des ensembles transfinis. Par G. Cantor. Traduction de F. Marotte. (Paris: Hermann, 1899, 97 pp.) Cantor's memoir, which appeared as two articles in the *Mathematischen Annalen*, Bde. 46-49, is a systematic introduction to his theory of transfinite—that is to say (§ 6) not finite—numbers. Two assemblages of elements are defined to be equivalent if a one-to-one correspondence can be instituted between the elements of one and the elements of the other. The "cardinal number" or *puissance* (hardly a happy translation of Mächtigkeit) of the assemblage is obtained by taking all the elements to be units. The properties of sums, products, and powers of the cardinal numbers (finite or transfinite) are then established.

The smallest transfinite number, viz. that which corresponds to the assemblage of all integers, is symbolized by $\aleph_0$ (alef-zero), and it is shown that any integral algebraical expression as $\aleph_0$ with finite coefficients and indices is equal to $\aleph_0$. To obtain greater transfinite numbers the notion of types (*Ordnungstypen*) is introduced. A type is an assemblage, of which the elements are arranged in a definite order. As examples may be mentioned the assemblage of positive proper fractions arranged in ascending order of magnitude; in another type with the same

elements p/q follows r/s if $p+q>r+s$, or if $p+q=r+s$ and $p>r$. The latter type has an initial element, the former has not. The transfinite number next higher than $\aleph_0$ is the number of types, subject to some conditions, of $\aleph_0$ elements, and is denoted by $\aleph_1$.

In spite of the clear presentation of the proofs of the several theorems we cannot help thinking that a Minute philosopher of to-day would find difficulties in the theory; probably about the *sub audita*. T.

New Plane and Solid Geometry. By W. W. BEMAN and D. E. SMITH. (Ginn & Co., Boston, U.S.A., 1899, ix + 382 pages.) This book is an attempt to give the student a knowledge of Euclidean and the more elementary modern geometry without following Euclid's order. The preface states that the author's object has been to give the minimum number of proofs and the maximum of exercises. Many of Euclid's propositions have been omitted, but formal proofs have been given of a number of very elementary propositions—not given by Euclid—which might with advantage have been treated as exercises. The early introduction of names and propositions belonging to modern geometry throws an unnecessary burden on the student's memory, as also does the use of such terms as *prismatic, pyramidal, and cylindrical space*. One feature of this book that might with advantage be introduced into all elementary books on geometry is an *Index of Etymologies*. A. W. S.

The Elements of Euclid. By I. TODHUNTER, D.Sc., F.R.S. New Edition, Revised and Enlarged, by S. L. LONEY, M.A., 4s. 6d. (Macmillan.) Todhunter's Euclid was well worth revision. "The text of the propositions has been simplified and shortened and . . . some of the proofs have been altered in accordance with modern usage." Each proposition is commenced on a fresh page, and the notes and exercises are now attached to the propositions to which they refer. Additional sections treat of the usual elementary developments of modern geometry. The proofs of Book V. are shortened, and to them as well as to those of Book II. the algebraical formulae are added. An attractive text-book.

Algebra Elementare. By MARCO NASSO. (Torino.) This is an excellent book for beginners, more after the English style than most Continental text-books, inasmuch as the sections are followed by examples, in all 2000 in number. There appear, however, to be no answers. A large number of examples are worked out in the text, and notes explanatory of difficulties are lavishly given. An interesting feature of the book is the number of purely historical notes. Professor Nasso is to be congratulated on having produced the best elementary Italian text-book we have seen.

Problèmes de Géométrie élémentaire, groupés d'après les méthodes à employer pour leur résolution. By I. ALEXANDROFF. Traduit du russe sur la 6me. édition. (Paris: Hermann, 1899, 5 fr. In 8°, pp. xii. + 154.) The main difference between Petersen's *Méthodes et Théories* and this Russian Geometry is the larger number of typical and miscellaneous examples. The section on rotation round a point (p. 107) might be more fully treated, and examples on inverse similitude should be added in the 7th edition. The application of algebra to the geometry of space receives but meagre treatment, while the author merely flirts with "the application of trigonometry to geometrical problems." But this volume will, we think, be more useful to the teacher for class purposes than Petersen's.

An Elementary Treatise on Practical Mathematics, for Technical Colleges and Schools. By J. GRAHAM, B.A., B.E. 3s. 6d. (Pp. iv. + 276. E. Arnold.) A manual restricted to elementary analysis, with special attention to practical applications and with deliberate avoidance of theoretical exposition. There are good chapters on plotting of curves, approximate methods of finding plane areas (Strip and Simpson's rule), with all the necessary apparatus of tables of logs., antilogs., etc. In the somewhat scrappy sections dealing with elementary algebra, we notice with pleasure the solution of simultaneous equations by graphs and by determinants. The student should not accept without question § 69-73. Useful exercises in English Grammar might be compiled from this manual.

BOOKS, MAGAZINES, ETC., RECEIVED.

The American Mathematical Monthly. Vol. VI., No. 3. March, 1896. Edited By Professor B. F. FINKEL and Professor M. COLAW.

Sulla Definizione del Numero. By Professor R. BETTAZI. (Torino, pp. 1-8.)

Algebra Elementare. By Professor MARCO NASSO. (Torino, 1898.)

Systems of Circles analogous to Tucker Circles. By J. A. THIRD, M.A. (Proc. Edin. M. Soc., Vol. XVII., 1898-99.)

The Elements of Euclid. By I. TODHUNTER, D.Sc., F.R.S. New Edition, Revised and Enlarged, by S. L. LONEY. (Pp. 332. i.—cxxxii. Macmillan & Co. 4s. 6d. 1899.)

An Elementary Treatise on Practical Mathematics. By JOHN GRAHAM, B.A., B.E. (Pp. 276. 3s. 6d. E. Arnold, 1899.)

Problèmes de Géométrie Elémentaire. Par IVAN ALEXANDROFF. Traduit du russe, sur la 6ᵉ édition. (Paris: Hermann. 5 fr.)

Urkunden zur Geschichte der Nichteuklidischen Geometrie, von F. Engel und P. Staeckel. I. Nikolai Ivanovitsch Lobatschefski. By Professor G. B. HALSTED. (Science N.S., Vol. IX., No. 232; pp. 813-817. June, 1899).

Le Mois Scientifique et Industriel. (Paris. 1.25 fr. Pp. 1-80.)

Periodico di Matematica. May—August, 1899. Edited by Professor DOTT. G. LAZZERI. (Livorno.)

Supplemento al Per. di Mat. VIII.-IX. May—June, 1899.

Il Pitagora, Anno V. (1° Sem.), 5-6. May—August, 1899. Edited by Prof. FAZZARI. (Palermo.)

The American Mathematical Monthly. VI., 3. March, 1899. Edited by Prof. B. F. FINKEL, A.M., M.Sc., and J. M. COLAW, A.M., Drury College, Springfield, Mo., U.S.A.

Bulletin de la Société Physico-Mathématique de Kasan. Second Series. VIII. 4, IX., 1-2. (Kasan, 1899.)

Journal de Mathématiques Elémentaires. April—August, 1899. Edited by Prof. G. MARIAUD. (Delagrave, Paris.)

Proceedings of the Edinburgh Mathematical Society. Edited by W A. LINDSAY, M.A., C.G., KNOTT, D.Sc., and C. TWEEDIE, M.A. Vol. XVII. (7s. 6d. Pp. 1-112. Williams & Norgate.)

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